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Noether's theorem in the nonlocal field theories

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Abstract

The explicit expressions for the vector current of internal symmetries, the density of an energy-momentum tensor and the density of an angular momentum tensor have been obtained for the field theories with the higher order derivatives and the nonlocal field theories. As an illustration of obtained results the simple example of the nonlocal theory of the charged scalar field had been studied. Finally, the simple closed expressions for the conserved quantities had been obtained.

1 Introduction

According to the Noether's theorem [1] the invariance of the total Lagrangian of a physical system with respect to the continuous group of transformations leads to the existence of the conserved quantities or the so called Noether charges. In the literature one can find the different field theories with the higher order derivatives (see, e.g. [2, 3]) or the theories containing the nonlocal interactions [4, 5]. There appears the question whether one can generalize the Noether's theorem to such cases?

As an initial step, let us consider the Lagrangian which contains along with the field Ψ also it's higer order derivatives $\partial_{\mu_1} \ldots \partial_{\mu_n} \Psi$ up to the order of $n \geq 1$ in the form

$$\mathcal{L} = \mathcal{L}(\Psi, \Psi_{,\mu_1}, \ldots, \Psi_{,\mu_1 \ldots \mu_n}). \tag{1}$$

The Lagrangian (1) is still local since it is a function of the field and its finite order derivatives evaluated at a single point of the spacetime. In order to satisfy to the nonlocal field theory one needs to include in (1) the dependence on the infinite number of field derivatives, i.e. tend $n \to \infty$.

Hereafter here we use the system of units in which $\hbar = 1 = c$. Indices $\mu, \nu, ...$ are taken to run from 0 to 3. Indices $\alpha, \beta, ...$ denote spatial components of corresponding quantities and run from 1 to 3. We use the time-like signature $(+, -, -, -)$ for the metric tensor $g_{\mu\nu}$.

2 Symmetries and the conserved quantities

All the observable quantities can be expressed in terms of the fields and their combinations. The fields included in Lagrangian, in general, belong to the representation space of the internal group of symmetry. Linear transformations of the fields related to the internal symmetry group do not violate the physical quantities, which is the case consedered in the present report. Thus, for the infinitesimal transformations related to the internal symmetry one can write the transformation matrix in the form close to unity:

$$U(\omega) = 1 - i\omega^a T^a, \tag{2}$$

where ω^a is a set of infinitesimal real parameters and T^a is a generator matrix. If the matrix U is unitary, then T^a are Hermitian matrices, and e.g. for the symmetry group $U(1)$, $T^a = 1$, and for $SU(2)$, T^a are the Pauli matrices.

In the most general case, along with the internal symmetry of system one needs to take into account the existence of the external symmetries related to the invariance of the physical quantities with respect to the translations of coordinates to some vector b^μ and the Lorentz transformations. The invariance under the spacetime translations leads to the conservation of the energy-momentum, while the rotational invariance or the invariance under the Lorentz transformations gives rise to the conservation of an angular momentum. For the infinitesimal Lorentz transform the transformation of coordinates can be realized by the matrix $a^\mu_\nu = \delta^\mu_\nu + \varepsilon^\mu_\nu$, where ε^μ_ν is an infinitesimal antisymmetric tensor. This implyies, that the infinitesimal Lorentz transformation matrix can be written in the following form

$$S(a) = 1 - \frac{i}{2}\varepsilon_{\mu\nu}\Sigma^{\mu\nu}, \tag{3}$$

where $\Sigma^{\mu\nu}$ is a Hermitian matrix which is in accordance with the field transformations. Thus, the full transformation of the field corresponding to the internal and external symmetries can be expressed in the form of matrices as follows

$$\Psi'(x') \equiv \Psi'(ax + b) = US(a)\Psi(x), \tag{4}$$

where the notation taken in the second parentheses of Eq.(4) fixes a particular order of transformations. Namely, the translation is performed after the Lorentz transformation. In the opposite case, one needs to use $x' = a(x + b)$. However, in particular case of a scalar field, by the definition we obtain the simple expression $\phi'(x') = \phi(x)$, where ϕ is a scalar with respect to the internal group of symmetries and with respect to the Lorentz transformations. The general $\Psi(x)$ belongs to some nontrivial representation space of the group of internal symmetries and the Lorentz group.

For the infinitesimal quantities ω^a, $\varepsilon^{\mu\nu}$ and b^μ the variation of the field takes the form

$$\begin{aligned}\delta\Psi(x) &= \Psi'(x)-\Psi(x)\\ &= S(a)U(\omega)\Psi(a^{-1}x-b)-\Psi(x)\\ &= (-i\omega^a T^a)\,\Psi(x)-b^\mu\partial_\mu\Psi(x)+\frac{1}{2}\varepsilon^{\mu\nu}\left(x_\mu\partial_\nu\Psi(x)-x_\nu\partial_\mu\Psi(x)-i\Sigma_{\mu\nu}\Psi(x)\right). \quad (5)\end{aligned}$$

Here we suppose that the initial transformations occur due to the matrix $U(\omega)$, then we make the Lorentz transformations and the translation. However the order of matrices $S(a)$ and $U(\omega)$ can be rearranged since the transformations of internal symmetries commute with the external symmetry transformations. Thus, the first term in the right hand side of Eq.(5) correspond to the transformations of internal symmetries, second to the translations and the third to the Lorentz transformations.

Reverting to Eq.(1) for the infinitesimal quantities ω^a, $\varepsilon^{\mu\nu}$ and b^μ one can write now the variation of the Lagrangian $\delta\mathcal{L}(x)=\mathcal{L}'(x)-\mathcal{L}(x)$ as

$$-b^\sigma\partial_\sigma\mathcal{L}-\varepsilon^{\sigma\nu}x_\nu\partial_\sigma\mathcal{L}=\frac{\partial\mathcal{L}}{\partial\Psi}\delta\Psi+\sum_{n\geq 1}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\dots\partial_{\mu_n}\Psi)}\partial_{\mu_1}\dots\partial_{\mu_n}\delta\Psi. \quad (6)$$

In order to derive an equation for the Noether current one needs to use the generalized higher order Euler-Lagrange equation which is sometimes called Euler-Poisson equation:

$$\frac{\partial\mathcal{L}}{\partial\Psi}+\sum_{n\geq 1}(-)^n\partial_{\mu_1}\dots\partial_{\mu_n}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\dots\partial_{\mu_n}\Psi)}=0. \quad (7)$$

Replacing the first term on the right side of Eq.(6) by the expression from the Euler-Lagrange equation (7), one can rewrite the right side of the Eq.(6) as follows

$$-\sum_{n\geq 1}(-)^n\partial_{\mu_1}\dots\partial_{\mu_n}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\dots\partial_{\mu_n}\Psi)}\delta\Psi+\sum_{n\geq 1}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\dots\partial_{\mu_n}\Psi)}\partial_{\mu_1}\dots\partial_{\mu_n}\delta\Psi. \quad (8)$$

The purpose is to present the above expression in the form of the divergence of some quantity. The term of the order n under the summation symbol in the first term of (8) can be rewritten in the form

$$\begin{aligned}&-(-)^n\partial_{\mu_1}\dots\partial_{\mu_n}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\dots\partial_{\mu_n}\Psi)}\delta\Psi\\ &=-(-)^n\partial_{\mu_1}\left(\partial_{\mu_2}\dots\partial_{\mu_n}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\dots\partial_{\mu_n}\Psi)}\delta\Psi\right)+(-)^n\partial_{\mu_2}\dots\partial_{\mu_n}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\dots\partial_{\mu_n}\Psi)}\partial_{\mu_1}\delta\Psi. (9)\end{aligned}$$

The first term on the right side has the form of divergence, while in the second term the derivative ∂_{μ_1} is shifted to the right and acts to $\delta\Psi$. Rewriting the second term of (9) in the same way

$$\begin{aligned}&(-)^n\partial_{\mu_2}\dots\partial_{\mu_n}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\dots\partial_{\mu_n}\Psi)}\partial_{\mu_1}\delta\Psi\\ &=(-)^n\partial_{\mu_2}\left(\partial_{\mu_3}\dots\partial_{\mu_n}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\dots\partial_{\mu_n}\Psi)}\partial_{\mu_1}\delta\Psi\right)-(-)^n\partial_{\mu_3}\dots\partial_{\mu_n}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\dots\partial_{\mu_n}\Psi)}\partial_{\mu_1}\partial_{\mu_2}\delta\Psi, (10)\end{aligned}$$

we obtain again the divergence in the first term and one more derivative in the second term. This implyies, that using such a recursion one can throw the derivative to the right side before

$\delta\Psi$. At each such a procedure the remaining term changes its sign and the rest term has a form of the divergence. Finally, the last term of recursion can be obtained by throwing over n derivatives which will have an additional sign $(-1)^n$ and consequently, its sign become opposite to the sign of the second term of Eq.(8) and it vanishes.

The result of this procedure for the expression uder summation of the first term of (8) has the form

$$
\begin{aligned}
&-(-)^n\partial_{\mu_1}\dots\partial_{\mu_n}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\dots\partial_{\mu_n}\Psi)}\delta\Psi+\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\dots\partial_{\mu_n}\Psi)}\partial_{\mu_1}\dots\partial_{\mu_n}\delta\Psi\\
&=\sum_{k=1}^{n}(-)^{n+k}\partial_{\mu_k}\left(\partial_{\mu_{k+1}}\dots\partial_{\mu_n}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\dots\partial_{\mu_n}\Psi)}\partial_{\mu_1}\dots\partial_{\mu_{k-1}}\delta\Psi\right)\\
&=\partial_\sigma\sum_{k=1}^{n}(-)^{n+k}\left(\partial_{\mu_{k+1}}\dots\partial_{\mu_n}\frac{\partial\mathcal{L}}{\partial(\partial_\sigma\partial_{\mu_2}\dots\partial_{\mu_n}\Psi)}\partial_{\mu_2}\dots\partial_{\mu_k}\delta\Psi\right). \qquad (11)
\end{aligned}
$$

Thus, Eq.(6) can be written in the form of the total divergence as follows

$$
\partial_\sigma\left[\sum_{n\geq1}\sum_{k=1}^{n}(-)^{n+k}\left(\partial_{\mu_{k+1}}\dots\partial_{\mu_n}\frac{\partial\mathcal{L}}{\partial(\partial_\sigma\partial_{\mu_2}\dots\partial_{\mu_n}\Psi)}\partial_{\mu_2}\dots\partial_{\mu_k}\delta\Psi\right)+b^\sigma\mathcal{L}+\varepsilon^{\sigma\nu}x_\nu\mathcal{L}\right]=0, \qquad (12)
$$

where $\delta\Psi$ in the parentheses is given by Eq.(5). The terms linear by the parameter ω^a determine the set of conserved currents $\mathfrak{J}^{a\sigma}$ connected with the internal group of symmetry. The terms proportional to the vector $-b^\sigma$ determine the conserved second-rank tensor which can be identified with the energy-momentum tensor $\mathfrak{T}_\mu^\sigma$. And finally the terms proportional to the tensor $\varepsilon^{\mu\nu}$ determine the conserved third-rank tensor $\mathfrak{M}_{\mu\nu}^\sigma$. The spatial components of this tensor correspond to the density of the total angular momentum of the system. In the case with $n=1$ we obtain the standard result (see, e.g. [6]). Thus the set of conserved quantities take the form

$$
\mathfrak{J}^{a\sigma}=\sum_{n\geq1}\sum_{k=1}^{n}(-)^{n+k}\left(\partial_{\mu_{k+1}}\dots\partial_{\mu_n}\frac{\partial\mathcal{L}}{\partial(\partial_\sigma\partial_{\mu_2}\dots\partial_{\mu_n}\Psi(x))}\right)\partial_{\mu_2}\dots\partial_{\mu_k}(-iT^a)\Psi(x), \qquad (13)
$$

$$
\mathfrak{T}_\mu^\sigma=\sum_{n\geq1}\sum_{k=1}^{n}(-)^{n+k}\left(\partial_{\mu_{k+1}}\dots\partial_{\mu_n}\frac{\partial\mathcal{L}}{\partial(\partial_\sigma\partial_{\mu_2}\dots\partial_{\mu_n}\Psi(x))}\right)\partial_\mu\partial_{\mu_2}\dots\partial_{\mu_k}\Psi(x)-\delta_\mu^\sigma\mathcal{L}, \qquad (14)
$$

$$
\begin{aligned}
\mathfrak{M}_{\mu\nu}^\sigma&=\sum_{n\geq1}\sum_{k=1}^{n}(-)^{n+k}\left(\partial_{\mu_{k+1}}\dots\partial_{\mu_n}\frac{\partial\mathcal{L}}{\partial(\partial_\sigma\partial_{\mu_2}\dots\partial_{\mu_n}\Psi(x))}\right)\partial_{\mu_2}\dots\partial_{\mu_k}\\
&\times\left(x_\mu\partial_\nu-x_\nu\partial_\mu-i\Sigma_{\mu\nu}\right)\Psi(x)-\left(x_\mu\delta_\nu^\sigma-x_\nu\delta_\mu^\sigma\right)\mathcal{L}. \qquad (15)
\end{aligned}
$$

The Noether's theorem allows us to find the conserved currents accurate within the arbitrary factor. In Eqs.(13)-(15) the factors are chosen in such a way, that the quantity $\mathfrak{T}_0^0$ coincides with the energy density of the system, which is defined by the Legendre transformations of the given Lagrangian. The quantity $\mathfrak{M}_{\alpha\beta}^0$, then, coincides with the density of an angular momentum of the system for the spatial indices α and β.

3 Nonlocal Lagrangian of the relativistic scalar charged particle

In the nonlocal field theory one should expand the nonlocal operators into series in differential operators as it is demonstrated in the next example.

Let us consider the Lagrangian which describes the relativistic scalar charged particle with mass m:

$$\mathcal{L} = \phi^* \left(i\partial_t - \sqrt{-\Delta + m^2} \right) \phi. \tag{16}$$

The particles have a relativistic dispersion law $E(p) = \sqrt{p^2 + m^2}$, however due to absence of the complex conjugate negative spot solution the particles do not have the corresponding antiparticles. The problem has a methodological and historical interest so far as the Lagrangian (16) had been considered in the past as the possible generalisation of the Schrödinger equation.

The Lagrangian (16) is invariant under the phase rotation of ϕ, which implyies the existence of the conserved vector current. Expanding the Lagrangian $\mathcal{L}$ in power series one can rewrite Eq.(16) in the following form

$$\mathcal{L} = \phi^* i\partial_t \phi - \sum_{l=0}^{\infty} f_l(m) \phi^* \Delta^l \phi, \tag{17}$$

where

$$f_l(m) = (-1)^l \frac{\Gamma(\frac{3}{2})\ m}{l!\ \Gamma(\frac{3}{2} - l)\ m^{2l}}. \tag{18}$$

One can easily find a zero component of the conserved quantities, as the Lagrangian (17) contains only the first derivative with respect to time. This implyies, that the series in Eqs.(13)-(15) terminate on the first term of the sum. Thus, the densities of the charge $\mathfrak{J}^0$, energy $\mathfrak{T}^0_\mu$ and angular momentum $\mathfrak{M}^0_{\mu\nu}$ take the following simple form

$$\mathfrak{J}^0 = \phi^* \phi, \tag{19}$$

$$\mathfrak{T}^0_\mu = \phi^* \partial_\mu \phi, \tag{20}$$

$$\mathfrak{M}^0_{\mu\nu} = \phi^* \left(x_\mu \partial_\nu - x_\nu \partial_\mu \right) \phi \tag{21}$$

Let us find now the spatial components of the conserved quantities. In order to do that one needs to specify the definition of the derivatives in the expressions (13)-(15). The derivative

$$\frac{\partial \mathcal{L}}{\partial(\partial_{\mu_1} \partial_{\mu_2} \ldots \partial_{\mu_n} \Psi)}$$

is symmetric under the permutation of indices. In order to calculate the spatial components of the conserved quantities we use the following general formulae

$$\frac{\partial}{\partial(\partial_\mu \Psi)} \partial_\tau \Psi = \delta^\mu_\tau, \tag{22}$$

$$\frac{\partial}{\partial(\partial_\mu \partial_\nu \Psi)} \partial_\tau \partial_\sigma \Psi = \delta^{(\mu}_\tau \delta^{\nu)}_\sigma, \tag{23}$$

$$\frac{\partial}{\partial(\partial_{\mu_1} \partial_{\mu_2} \ldots \partial_{\mu_n} \Psi)} \partial_{\nu_1} \partial_{\nu_2} \ldots \partial_{\nu_n} \Psi = \delta^{(\mu_1}_{\nu_1} \delta^{\mu_2}_{\nu_2} \ldots \delta^{\mu_n)}_{\nu_n}. \tag{24}$$

In particlar,

$$\frac{\partial}{\partial(\partial_{\mu_1}\partial_\nu\Psi)}\Box\Psi = g^{\mu\nu}. \tag{25}$$

The indices in the parentheses are symmetrized and reflect the symmetry of the quantities $\partial_{\mu_1}\partial_{\mu_2}\dots\partial_{\mu_n}\Psi$ with respect to the permutation of indices. Thus, e.g. the tensor $\delta^{(\mu_1}_{\nu_1}\delta^{\mu_2}_{\nu_2}\dots\delta^{\mu_n)}_{\nu_n}$ contains in total $n!$ of possible permutations of combinations of $\delta^{\mu_1}_{\nu_1}\delta^{\mu_2}_{\nu_2}\dots\delta^{\mu_n}_{\nu_n}$:

$$\delta^{(\mu_1}_{\nu_1}\delta^{\mu_2}_{\nu_2}\dots\delta^{\mu_n)}_{\nu_n} = \frac{1}{n!}\sum_{\sigma(i_1 i_2 \dots i_n)}\delta^{\mu_{i_1}}_{\nu_1}\delta^{\mu_{i_2}}_{\nu_2}\dots\delta^{\mu_{i_n}}_{\nu_n}, \tag{26}$$

where $\sigma(i_1 i_2 \dots i_n)$ denotes all the possible permutations of the numbers $(1, 2, \dots, n)$.

In our particular case given by Eq.(17), the Lagrangian depends on the even order derivatives. This implyies, that the expression $\Box^l\Psi = (\partial_{\mu_1}\partial_{\mu_2}\dots\partial_{\mu_{2l}}\Psi)\, g^{\mu_1\mu_2}\dots g^{\mu_{2l-1}\mu_{2l}}$ for $n = 2l$ can be written as

$$\frac{\partial}{\partial(\partial_{\mu_1}\partial_{\mu_2}\dots\partial_{\mu_{2l}}\Psi)}\Box^l\Psi = g^{(\mu_1\mu_2}g^{\mu_3\mu_4}\dots g^{\mu_{2l-1}\mu_{2l})}. \tag{27}$$

The right side of Eq.(27) can be rewritten in the form similar to (26)

$$g^{(\mu_1\mu_2}g^{\mu_3\mu_4}\dots g^{\mu_{2l-1}\mu_{2l})} = \frac{2^l l!}{(2l)!}\sum_{\text{diff. pairs}} g^{\mu_1\mu_2}g^{\mu_3\mu_4}\dots g^{\mu_{2l-1}\mu_{2l}}. \tag{28}$$

Equation (27) comes from the tensor contraction of (26) with the product of $g^{\mu_1\mu_2}\dots g^{\mu_{2l-1}\mu_{2l}}$. In the contraction each of the combinations of the products of metric tensors contains $2^l l!$ times. This factor is taken outside the sum in Eq.(28). In addition, the summation operates in all $2l$ pair of indices. This implyies, that the total number of such pairs is $(2l)!/(2^l l!)$.

Multiplying Eq.(27) to $\partial_{\mu_1}\partial_{\mu_2}\dots\partial_{\mu_{2l}}\Psi$ and summing up over the indices one obtains

$$\begin{aligned}
\partial_{\mu_1}\partial_{\mu_2}\dots\partial_{\mu_{2l}}\Psi\frac{\partial}{\partial(\partial_{\mu_1}\partial_{\mu_2}\dots\partial_{\mu_{2l}}\Psi)}\Box^l\Psi &= \partial_{\mu_1}\partial_{\mu_2}\dots\partial_{\mu_{2l}}\Psi g^{(\mu_1\mu_2}g^{\mu_3\mu_4}\dots g^{\mu_{2l-1}\mu_{2l})} \\
&= \partial_{\mu_1}\partial_{\mu_2}\dots\partial_{\mu_{2l}}\Psi g^{\mu_1\mu_2}g^{\mu_3\mu_4}\dots g^{\mu_{2l-1}\mu_{2l}} \\
&= \Box^l\Psi.
\end{aligned} \tag{29}$$

The expression $\partial_{\mu_1}\partial_{\mu_2}\dots\partial_{\mu_{2l}}\Psi$ is symmetric under the permutation of indices which leads to the vanishing of the symmetrization sign (superscript parantheses) in the products of the metric tensors $g^{\mu\nu}$, bringing up a simple result (29). It is a standard way of the constraction of the Lagrangian. The above equations can be rewritten for the three-dimensional space by the replacement $\Box \to \Delta,\ \ g^{\mu\nu} \to \delta^{\alpha\beta}$.

Taking into account the discussion above we find the spatial components of the conserved current in the form

$$\mathfrak{J}^\alpha = -\sum_{l=1}^{\infty} f_l(m)\sum_{k=1}^{2l}(-)^k\left(\partial_{\alpha_{k+1}}\dots\partial_{\alpha_{2l}}\phi^*\delta^{(\alpha\alpha_2}\delta^{\alpha_3\alpha_4}\dots\delta^{\alpha_{2l-1}\alpha_{2l})}\partial_{\alpha_2}\dots\partial_{\alpha_k}(-i)\phi\right), \tag{30}$$

where $f_l(m)$ is given by Eq.(18) and $\delta\phi = -i\phi$ for the symmetry group $U(1)$. The same analysis leads to the conservation of the spatial components of the densities of the energy-momentum and angular momentum tensors. Taking into account $\delta^\alpha_\mu\mathcal{L} = 0$, one can rewrite Eqs.(14) for the Lagrangian (17) in the form

$$\mathfrak{T}^\alpha_\mu = -\sum_{l=1}^{\infty} f_l(m) \sum_{k=1}^{2l} (-)^k \left(\partial_{\alpha_{k+1}} \dots \partial_{\alpha_{2l}} \phi^* \delta^{(\alpha\alpha_2}\delta^{\alpha_3\alpha_4} \dots \delta^{\alpha_{2l-1}\alpha_{2l})} \partial_\mu \partial_{\alpha_2} \dots \partial_{\alpha_k} \phi \right). \quad (31)$$

The equation (15) for the density of angular momentum can be expressed in terms of (31). Taking $\Sigma_{\mu\nu} = 0$ for the charged scalar field and $U(1)$ symmetry group one can rewrite (15) in the form

$$\mathfrak{M}^\alpha_{\mu\nu} = x_\mu \mathfrak{T}^\alpha_\nu - x_\nu \mathfrak{T}^\alpha_\mu. \quad (32)$$

In the lowest order approximation, $f_1(m) = -1/(2m)$. Then Eqs.(30)-(32) reduce to the standard expressions

$$\mathfrak{J}^\alpha = \frac{1}{2m} \left\{ (i\partial^\alpha \phi^*)\phi + \phi^*(-i\partial^\alpha \phi) \right\} + \dots, \quad (33)$$

$$\mathfrak{T}^\alpha_\mu = \frac{1}{2m} \left\{ (-\partial^\alpha \phi^*)(\partial_\mu \phi) + \phi^*(\partial_\mu \partial^\alpha \phi) \right\} + \dots, \quad (34)$$

$$\mathfrak{M}^\alpha_{\mu\nu} = \frac{1}{2m} \left\{ (-\partial^\alpha \phi^*)(x_\mu \partial_\nu \phi - x_\nu \partial_\mu \phi) + \phi^*(x_\mu \partial_\nu \partial^\alpha \phi - x_\nu \partial_\mu \partial^\alpha \phi) \right\} + \dots. \quad (35)$$

The series in Eqs.(30)-(32) can be summarized using the general factorization formula

$$a^n - b^n = (a-b) \sum_{k=0}^{n-1} a^{n-1-k} b^k. \quad (36)$$

we turn to the momentum space substituting in Eqs.(30)-(32) the fields normalized to a plane wave with the momenta p' и p. The four-momentum operator in coordinate space is given by

$$p^\mu = (E, p^\alpha) = i\left(\frac{\partial}{\partial x_\mu}, -\nabla \right) \quad (37)$$

Subsituting (36) into (30) and changing in the series, n to $2l$ with the shift of k to $k-1$ we obtain the transitional current

$$\mathfrak{J}^\alpha(p', p) = \frac{(p'+p)^\alpha}{E(p') + E(p)}, \quad (38)$$

and spatial components of energy-momentum tensor

$$\mathfrak{T}^\alpha_\mu(p', p) = \frac{(p'+p)^\alpha}{E(p') + E(p)} (p'+p)_\mu. \quad (39)$$

The angular momentum tensor, however, contains along with the dependence on p' and p also the dependence on the coordinates x_μ. Due to this reason the correct form of the equation would contain the action of the coordinate operator on the fields ϕ^* and ϕ. Keeping in mind this, for the elegance of the form of equation, formally, one can write

$$\mathfrak{M}^\alpha_{\mu\nu}(p', p) = \frac{(p'+p)^\alpha}{E(p') + E(p)} \left[x_\mu (p'+p)_\nu - x_\nu (p'+p)_\mu \right]. \quad (40)$$

Using Eqs.(19)-(21) one can write then zero components of the conserved quantities in the form

$$\mathfrak{J}^0(p', p) = 1, \quad (41)$$

$$\mathfrak{T}^0_\mu(p', p) = (p' + p)_\mu, \quad \mathfrak{T}^0_0 = E(p') + E(p), \tag{42}$$

$$\mathfrak{M}^0_{\mu\nu}(p', p) = \left[x_\mu(p' + p)_\nu - x_\nu(p' + p)_\mu\right], \quad \mathfrak{M}^0_{00} = 0. \tag{43}$$

It is easy to show that the combination of equations (38) and (41) satisfies to the continuity equation

$$(p' - p)_\mu \mathfrak{J}^\mu(p', p) = 0, \tag{44}$$

which leads to the conservation of the current $\mathfrak{J}^\mu$. Similarly, the energy momentum and angular momentum tensors satisfy to

$$(p' - p)_\mu \mathfrak{T}^\mu_\nu(p', p) = 0, \tag{45}$$

$$(p' - p)_\mu \mathfrak{M}^\mu_{\alpha\beta}(p', p) = 0. \tag{46}$$

In the four-dimentional flat spacetime, Eq.(45) brings us four conserved quantities, one for each of the translations b^σ corresponding to the energy and three components of momentum. Equation (46) leads in general to the conservation of six charges. For the indices $\alpha, \beta = 1, 2, 3$ the Lorentz transformation is simply the rotation and the total angular momentum of the system corresponds to the three conservations of charges.

Let the action now is given in the form

$$S = \int d^4x d^4y \phi^*(x) G^{-1}(x - y)\phi(y), \tag{47}$$

where $G^{-1}(x-y)$ - is the arbitrary Lorentz-invariant function. Obviously, the transition current takes the form

$$\mathfrak{J}^\mu(p', p) = (p' + p)^\mu \frac{G^{-1}(p') - G^{-1}(p)}{p'^2 - p^2}, \tag{48}$$

where $G^{-1}(p)$ is the Fourier transform of $G^{-1}(x)$. On the mass surface $p'^2 = p^2 = m^2$ the current is conserved, which follows from (44). Outside the mass surface one can write

$$(p' - p)_\mu \mathfrak{J}^\mu(p', p) = G^{-1}(p') - G^{-1}(p). \tag{49}$$

Taking into account that the physical meaning of the function $G^{-1}(x - y)$ is the inverse propagator, Eq.(49) is equivalent to the Ward identity.

The proof of Eq.(48) is simpler if one returns to the expression (8). Expanding at first $G^{-1}(p)$ in a Taylor series in the vicinity of a point $p = 0$, one finds

$$G^{-1}(p) = \sum_l G^{-1(l)}(0) \frac{p^{2l}}{l!}. \tag{50}$$

The Lagrangian corresponding to the action (47) takes the form

$$\mathcal{L} = \sum_{l=0}^{\infty} \frac{(-)^l}{l!} G^{-1(l)}(0) \phi^* \Box^l \phi. \tag{51}$$

The contribution of the term of the order of l into the current divergence is

$$-i\frac{(-)^l}{l!} G^{-1(l)}(0) \left(-\left(\Box^l \phi^*\right)\phi + \phi^* \Box^l \phi\right). \tag{52}$$

In the momentum space this expression takes the following form

$$\begin{aligned} i\frac{1}{l!}G^{-1(l)}(0)\left(p'^{2l}-p^{2l}\right) &= (p'-p)_\mu(p'+p)^\mu i\frac{1}{l!}G^{-1(l)}(0)\left(p'^{2l-2}+p'^{2l-4}p^2+\ldots+p^{2l-2}\right) \\ &= (p'-p)_\mu(p'+p)^\mu i\frac{1}{l!}G^{-1(l)}(0)\frac{p'^{2l}-p^{2l}}{p'^2-p^2}. \end{aligned} \quad (53)$$

Summing over the all values of l and omitting the derivative $\partial_\mu = i(p'-p)_\mu$, we come back to the expression (48).

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