Joint Institute For Nuclear Research
Laboratory of Information Technology

Final Report on the Summer Student Program

Structure of Stationary Neutron Stars

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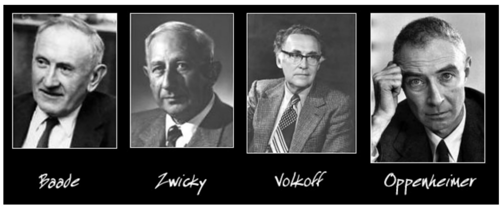
Abstract

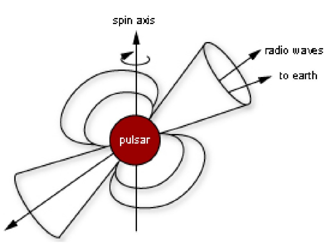
We start from the simplifying assumptions of staticity and spherical symmetry and show the derivation of the equations governing the structure of neutron stars. Derivations are made in the framework of the theory of general relativity. We make thermodynamic considerations to justify the way of introduction of the equation of state to the problem. With the use of appropriate equation of state we integrate the equations of structure to obtain the relevant parameters for the stars. In the end we give a brief discussion of the method of integration and present the results.

1 Introduction

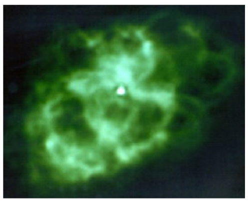
Neutron stars are stars of mass comparable to the mass of our Sun but with radii of only about 10 kilometres. Hence, neutron stars are very dense objects. Their density is of the same order as the density of atomic nucleus: $10^{-14}\,\mathrm{g\,cm^{-3}}$. They consist largely of nucleons most of them being neutrons. What supports these stars against gravity is the degeneracy pressure of neutron matter due to Pauli exclusion principle and nucleon nucleon interactions.



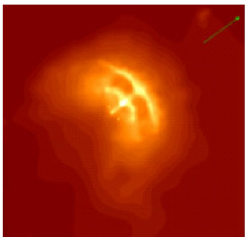
First ideas on the possibility of existence of such objects arose around the time neutron itself was discovered at beginning of 1930's. In December 1933, during a meeting of the American Physical Society at Stanford, William Baade and Fritz Zwicky predicted the existence of neutron stars as supernova remnants.



In 1960's it was argued that neutron stars could have very strong magnetic fields and after the pulsars were discovered in 1967 it was proposed that Pulsars are magnetized rotating neutron stars emitting a highly focused beam of electromagnetic radiation oriented long the magnetic axis. The misalignment between the magnetic axis and the spin axis leads to a lighthouse effect i.e. from Earth we see radio pulses.



(a) Crab Nebula in radio



(b) Pulsar in Vela Supernova in x-ray

Figure 1: Pulsars in remnants of supernovae

This hypothesis got its confirmation when pulsars were found in Crab Nebula and Vela Supernova remnant, the places where neutron stars were expected as was predicted some 35 years earlier.

One more problem of interest is the existence of maximum mass of a neutron star, or generally stars built of dense degenerate matter. Such demonstration was first done by Chandrasekhar for white dwarfs, another class of compact stars. Intuitively why a maximum mass should exist can be seen as follows: an increase in density leads to corresponding increase in gravitational attraction. To balance this increase an increment of pressure must be large enough. However the rate of change of pressure corresponding to the change in density is bounded. It is related to the speed of sound which cannot be greater than the speed of light. Hence at some point the increase in in density will not be balanced by increase of pressure which leads to the existence of a maximum mass of the star. In our calculations maximum mass shows itself to be around 2 solar masses.

2 Equations of Hydrostatic Equilibrium

In this section we derive the hydrostatic equations which in combination with adequate equation of state govern the structure of neutron stars. As we will see the equations must be derived in the framework of general theory of relativity. The comparison with Newtonian hydrostatic equations would show non negligible correction terms when massive objects of such small radius as neutron stars are treated.

We will here use the geometrized units(system of natural units in which the speed of light and gravitational constant are equal to unity). In this system of units Einstein field equations have the form:

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \tag{1}$$

We make assumption that the star is static, spherically symmetric and made of perfect fluid. The first two assumptions influence the choice of appropriate metric and thus lead to the calculation of the components of Einstein tensor, that is, the left-hand side of the equations (1). The second assumption of course determines the right-hand side of the equations since this becomes the stress-energy tensor of a perfect fluid.

2.1 Einstein tensor

The metric we use in our derivation is:

$$ds^2 = e^{2\Phi}dt^2 - e^{2\Psi}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2}$$

Φ and Ψ are functions of coordinate r only. It is not necessary to assume that there are no non-diagonal terms between t and r coordinates, this is rather a matter of freedom in choosing of the coordinates.

Let us instead of the coordinate basis chose an orthonormal system of which the basis 1-forms are:

$$\omega^0 = e^{\Phi}dt, \quad \omega^1 = e^{\Psi}dr, \quad \omega^2 = rd\theta, \quad \omega^3 = \sin\theta d\phi \tag{3}$$

In such basis the components of the metric tensor become equal to the components of the

Minkowsi metric:

$$g_{\mu\nu} = \eta_{\mu\nu} = diag(1, -1, -1, -1) \tag{4}$$

Now we need to calculate components of Einstein tensor. These are given by:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{5}$$

Where $R_{\mu\nu}$ is Ricci tensor, a contraction of the Riemann tensor $R_{\mu\nu} = \mathcal{R}^{\alpha}{}_{\mu\alpha\nu}$ and R is its trace. Riemann tensor can be written in terms of curvature 2-forms $\mathcal{R}_{\mu\nu}$ as follows:

$$\mathcal{R} = \mathcal{R}_{\mu\nu} \omega^{\mu} \wedge \omega^{\nu}, \tag{6}$$

while curvature 2-forms are given in terms of connection forms:

$$\mathcal{R}^{\mu}{}_{\nu} = d\omega^{\mu}{}_{\nu} + \omega^{\mu}{}_{\alpha} \wedge \omega^{\alpha}{}_{\nu} \tag{7}$$

All we need to do now is find the connection forms for our chosen basis. Connection forms are defined through the expression

$$d\omega^{\mu} = -\omega^{\mu}{}_{\nu} \wedge \omega^{\nu} \tag{8}$$

In our case basis 1-forms are of the form $\omega^{\mu} = f(\mu) dx^{\mu}$, where f in each case is a scalar function of the coordinates. Thus derivatives of basis 1-forms can be written as:

$$d\omega^{\mu} = df \wedge dx^{\mu} = f^{-1} df \wedge \omega^{\mu} \tag{9}$$

since $d(dx^{\mu}) = 0$. By comparing these expressions with expressions (8) we read off the connection forms. Using now the equation (7) we obtain curvature 2-forms from which we can deduce the components of Riemann tensor. We are now though only interested in contraction of this tensor which finally gives us components of Einstein tensor G. The non zero components found were:

$$\begin{aligned} G_{00} &= \frac{2e^{-2\Psi}}{r} \Psi' + \frac{1}{r^2}(1 - e^{-2\Psi}) \\ G_{11} &= \frac{2e^{-2\Psi}}{r} \Phi' - \frac{1}{r^2}(1 - e^{-2\Psi}) \\ G_{22} = G_{33} &= e^{-2\Psi}(\Phi'' + (\Phi')^2 - \Phi'\Psi' + \frac{1}{r}(\Phi' - \Psi')) \end{aligned} \tag{10}$$

More details on this derivation can be found for example in [1].

2.2 Stress-energy tensor

As we have already said we will treat the star as if it consists of matter that is a perfect fluid. The stress energy tensor for such a fluid is:

$$T = (\epsilon + p) u \otimes u - pg \tag{11}$$

where p is pressure an ϵ is energy density. The four-velocity u in our static case has its u_0 component equal to 1 and other components equal 0. Hence the stress-energy tensor is diagonal.

2.3 Solving the equations

We can now put the conclusions of the last two sections together into the equation (1). Three distinct equations are obtained:

$$\begin{aligned} 00: &\quad \frac{2e^{-2\Psi}}{r}\Psi' + \frac{1}{r^2}(1-e^{-2\Psi}) = 8\pi\epsilon \\ 11: &\quad \frac{2e^{-2\Psi}}{r}\Phi' - \frac{1}{r^2}(1-e^{-2\Psi}) = 8\pi p \\ 22: &\quad e^{-2\Psi}(\Phi'' + (\Phi')^2 - \Phi'\Psi' + \frac{1}{r}(\Phi' - \Psi')) = 8\pi p \end{aligned} \tag{12}$$

To solve the equations we first notice that G_{00} can be put in the form:

$$G_{00} = \frac{1}{r^2}(r(1-e^{-2\Psi}))' \tag{13}$$

so that we see that the 00-equation is integrable:

$$d(r(1-e^{-2\Psi})) = 8\pi r^2 \epsilon dr \tag{14}$$

The solution for Ψ is:

$$\Psi = -\frac{1}{2}\log(1-\frac{2m}{r}) \tag{15}$$

where we have defined $m = \int_0^r 4\pi r^2 \epsilon dr$. This we will call mass because the integral takes contribution of energy density inside a sphere of radius r(since we work in natural units, mass density and energy density are interchangeable). Substituting the expression (15) into 11-equation we get the expression for Φ':

$$\Phi' = \frac{m + 4\pi r^3 p}{r(r-2m)} \tag{16}$$

Substituting this expression and its derivative as well as (15) and its derivative into 22-equation we find:

$$p' + \Phi'(\epsilon + p) = 0 \tag{17}$$

Finally rewriting the last equation and taking into account (16) we arrive at the two equations that describe the hydrostatic equilibrium:

$$\begin{aligned} \frac{dp}{dr} &= -(\epsilon + p)\frac{m + 4\pi r^3 p}{r(r-2m)} \\ \frac{dm}{dr} &= 4\pi r^2 \epsilon \end{aligned} \tag{18}$$

These are the well known Tollmann-Openheimer-Volkoff equations first derived by J. Robert Oppenheimer and George Volkoff in their 1939 paper [6]. In order to integrate these equations energy density in terms of pressure must be known. This is a property of the matter at hand and is the topic of the next section. To conclude this section we will discuss initial values and conditions on boundary of the star that are also needed to provide a unique solution for the pressure, mass, energy density and metric coefficients. Obviously initial value for the mass is zero at the center of the star. While the central pressure is left as a free parameter. The pressure will be a monotonically decreasing function of r and will fall to zero at the surface of the star. The metric coefficient Φ is from the equation (16) given by the integral

$$\Phi = \Phi(0) + \int_0^r \frac{m + 4\pi r^3 p}{r(r-2m)} dr \tag{19}$$

On the outside of the star pressure is zero and the mass is constant, namely the total mass of the star $m(r) = M$. Than for $r > R$ where R is the radius of the star we have for Φ:

$$\Phi = \Phi(R) + \int_R^r \frac{M}{r(r-2M)} dr \tag{20}$$

This integral is now easily solved to give:

$$\Phi = \frac{1}{2}\log(1 - \frac{2M}{r}) + \Phi(R) - \frac{1}{2}\log(1 - \frac{2M}{R}) \tag{21}$$

At the distances far away from the star spacetime is flat and so when $r \to \infty$ we should be able to recover the Minkowski metric, that is Φ should be zero. From equation (21) when $r \to \infty$ we get:

$$\Phi(R) = \frac{1}{2}\log(1 - \frac{2M}{R}) \tag{22}$$

This is the value of metric coefficient Φ at the surface of the star and function $\Phi(r)$ obtained by integration of equation (16) with an arbitrary initial condition must be corrected to pass through this point.

3 Equation of state

The first law of thermodynamics states that:

$$dE = TdS - pdV + \mu dN \tag{23}$$

or written in terms of energy density ϵ, entropy density s and baryon number density n:

$$d\epsilon = Tds - p + \mu dn \tag{24}$$

So normally energy density is a function of entropy as well as baryon number density. Similarly we can write the Gibbs-Duhem relation in the same manner:

$$dp = Tds + nd\mu \tag{25}$$

from it we see that pressure is also function of baryon number and entropy density. However in the case of the neutron star thermal contributions to pressure and energy density are negligible and thus both temperature and entropy can be considered to be zero.

The temperatures of observed neutron stars are actually quite high, around $10^6\,\mathrm{K}$, but when dealing with the matter at such high densities degeneracy of the matter makes a much larger contribution. The temperature associated with the Fermi energy can be approximated by treating the neutron star matter as an ideal Fermi gas. The order of magnitude of the temperature in such approximation is $10^{11}\,\mathrm{K}$.[1][5]

Nuclear forces on the other hand do make a significant contributions that have to be included in a realistic equation of state of neutron star matter.

Since we have found that both pressure and energy density are functions of the baryon number density only:

$$p = p(n), \quad \epsilon = \epsilon(n) \tag{26}$$

this gives us the functional relationship between pressure and energy density we needed to integrate the hydrostatic equations(equations (18)).

The actual equation of state we use in our calculations is taken from [2] and is named HDD by the authors of that paper.

4 Numerical integration and results

Earlier in the paper we have expressed all the equations in natural units. For the actual calculations we wanted to have that the equations give us the values of pressure, energy density, mass and radius in units more appropriate to the problem at hand. These units for the pressure and energy density are: $\mathrm{MeV\,fm^{-3}}$. The mass we wanted expressed in terms of solar mass, $M_\odot$ and radius in kilometres. To obtain this we reintroduce the speed of light and gravitational constant to the equations (18) by changing from geometrized units to the units described above. The equations can then be put in the form:

$$\begin{aligned} \frac{dp}{dr} &= -\Gamma(\epsilon + p)\frac{m + ar^3 p}{r(r - 2\Gamma m)} \\ \frac{dm}{dr} &= ar^2\epsilon \end{aligned} \tag{27}$$

where conversion coefficients are:

$$\begin{aligned} \Gamma &= \frac{GM_\odot}{c^2} = 1.4765\,\mathrm{km} \\ a &= 4\pi\frac{\mathrm{km}^3}{M_\odot c^2}\frac{\mathrm{MeV}}{\mathrm{fm}^3} = 1.12659 \cdot 10^{-5} \end{aligned} \tag{28}$$

To integrate these equations numerically we have written a program in language c++. Integration over r was set to proceed until the pressure falls below zero. At that point the integration is stopped and correction to the last step is made by means of setting the pressure equal to zero and then interpolating radius at that point from the last three points in the integration.[3]

The equation of state used, the HDD equation, was provided to the program in form of the file with tabulated values of pressure, energy density, baryon number and chemical potential. In the program we implemented the equation of state as a function $\epsilon = \epsilon(p)$ shown in the Figure 2., with the intermediate values of energy density determined by interpolation.

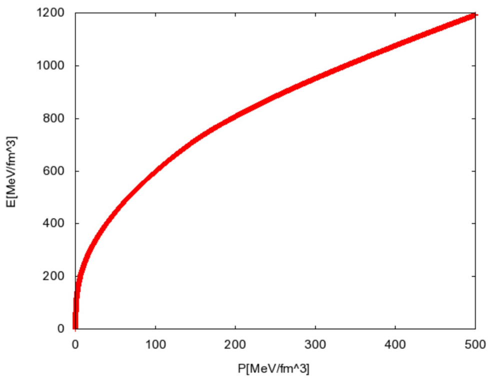


Figure 2: Energy density as a function of pressure for the HDD equation of state

The integration was done for many values of central pressure. Figure 3. shows how pressure changes inside of the star for one value of central pressure.

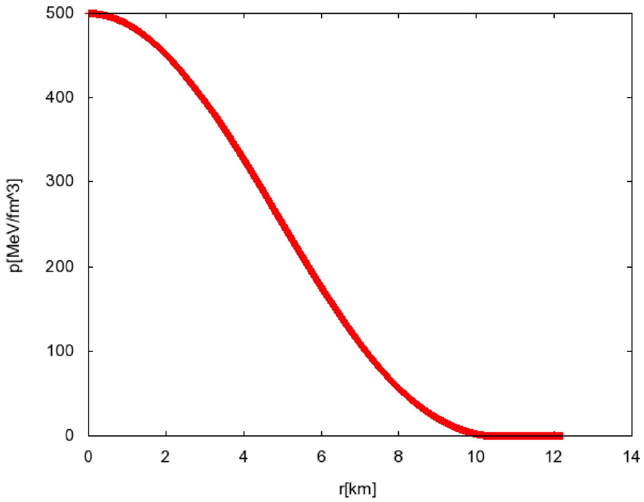


Figure 3: Pressure as a function of coordinate r for the central pressure of $500\frac{\text{MeV}}{\text{fm}^3}$

After each integration the radius and mass of the star were stored to produce the Figure 4. The figure shows the parametric plot of mass and radius for many values of pressure in the center of the star.

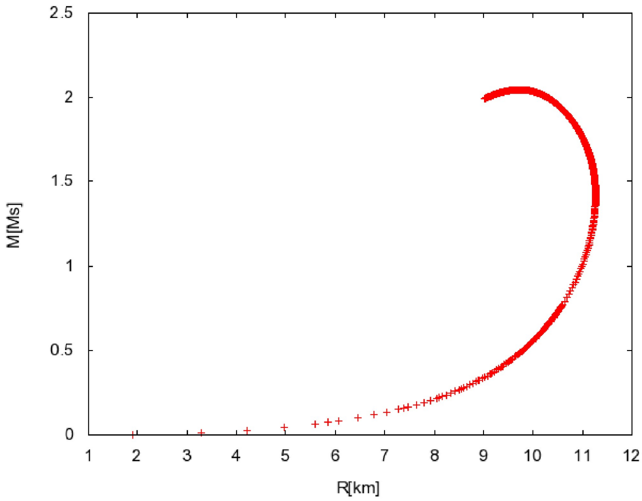


Figure 4: Mass versus radius plot for a wide range of central pressures

From the plot it can be seen that a maximum mass of the star with this equation of state is around $2M_\odot$, more precisely a value of $M = 2.048M_\odot$ was found. The radius of the star with maximum mass is around $9.7km$.

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