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The Laboratory of Information Technologies

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Movement of body in the gravitational field

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An overview of Work

Gravitaion is a manifestaion of spacetime curvature, and that curvature shows up in the veviation of one geodesic from a nearby geodics. The central issue: How can one quanify the "separtion"and the "rate of change"of "separion"of two "geodiscs"in "curved spacetime. Precise answer requires new concepts.
"Separation"between geodesics will mean "vector". But the concept of vector as employed in flat Lorentz spacetime must be sharpened up into the local concept of tangent vector, when on passes to curved spacetime.
Vectors in curved geometry need to define "separation"as to vector. Parallel transport in curved space time to compare separation vectors at neghboring points and to dedine the "rate of change of separation". The notion of parallel transport founds itself on the idea of "geodesic". We use geodesics to define parallel transport, uses parallel transport to define covartiant derivative, and uses covariant derivative to describe geodesics.
The main part of my work has become acquaintance with new mathematical aparatus, which was necessary for dealing with theme "Movement of body in the gravitational field". I had to improve my skills of differential geometry and topology, first part of my work was devoted to it.The use of studied theory in practice was the second part of this work.

Vector and directional derivative refined into tangent vector

Flat spacetime can accommodate several equivalent definitions of a vector, but other definitions of vector break down when metric is abandoned except vector definition as the "rate-of-change-of-point-along-curve". It is called "tangent vector". The tangent vector u to a curve $P(\lambda)$ is the directional derivative operator along that curve

$$u = \partial_\alpha = \left(\frac{d}{d\lambda}\right)$$

We can be choosen an event P_0. At that event choose any set of fout noncoplanar vectors e_0, e_1, e_2, e_3 - it is a basic on which to expend all other vectors at P_0:

$$u = u^\alpha e_\alpha, v = v^\alpha e_\alpha$$

Construct the four directional derivative operators $\partial_\alpha = \partial_{e_\alpha}$ along his four basic vectors. Then

$$\partial_u = u^\alpha \partial_\alpha, \quad \partial_v = v^\alpha \partial_\alpha$$

Hence,

$$\Rightarrow \;\; u = aw + bv \Leftrightarrow u^\alpha = aw^\alpha + bv^\alpha \Leftrightarrow \partial_u = a\partial_w + b\partial_v$$

This isomorphism extends to the concept "tangent space". Because linear relations among directional derivatives evaluated at one and the same point P_0 are meaningful and obey the usual addition and multiplication rules, these operators form an absrtact vector space called the tangent space at P_0

Bases, components, and transformation laws for vectors

An especially useful basis in the tangent space at an event P_0 is induced by any coordinate system$[x^0(P), x^1(P), x^2(P), x^3(P)]$:

$$e_0 = \left(\frac{\partial}{\partial x^0}\right), \; e_1 = \left(\frac{\partial}{\partial x^1}\right), \; e_2 = \left(\frac{\partial}{\partial x^2,}\right), \; e_3 = \left(\frac{\partial}{\partial x^3}\right)$$

A transformation from one basis to another in the tangent space is produced by a nonsingular matrix,

$$e_{\alpha'} = e_\beta L^\beta_{\alpha'}$$

and the components of a vector must tranform by the inverse matrix

$$u^{\alpha'} = L^{\alpha'}_\beta u^\beta$$

$$||L^{\alpha'}_\beta|| = ||L^\beta_{\gamma'}||^{-1}$$

This inverse tranformation law guarantees compatibility between the expansions $u = e_{\alpha'} u^{\alpha'} and\; u = e_\beta u^\beta$

$$u = e_{\alpha'} u^{\alpha'} = (e_\gamma L^\gamma_{\alpha'})(L^{\alpha'}_\beta u^\beta) = e_\gamma \delta^\gamma_\beta u^\beta$$

1-Forms

When the Lorentz metric is removed from spacetime one must sharpen up the concept of 1-from σ by insisting that it be attached to a specific event P in spacetime. Given any set of basis vectors e_0, e_1, e_2, e_3 at the event P, one construct the dual basis of 1-forms $\omega^0, \omega^1, \omega^2, \omega^3$ by choosing the surfaces of ω^β such that

$$< \omega^\beta, e_\alpha >= \delta^\beta_\alpha$$

A matvelously simple formalism for calculating and manipulating components of tangent vectors and 1-forms then results:

$$u = e_\alpha u^\alpha;\ \sigma = \sigma_\beta \omega^b$$

$$u^\alpha =< \omega^\alpha, u >;\ \sigma_\beta =< \sigma, e_\beta >$$

$$< \sigma, u >= \sigma_\alpha u^\alpha$$

$$\omega^{\alpha'} = L^{\alpha'}_\beta \omega^\beta;\ \ \sigma_{\alpha'} = \sigma_\beta L^\beta_{\alpha'}$$

In the absence of a metric, there is no way to pick a specific 1-form at an event P_0 and say that it corresponds to a specific tangent vector u at P_0.

df now resides in the tangent space tarher than in spacetime itself. There is no change in the fundamental equation relating the projection of the gradient to the directional derivative:

$$< df, u >= \partial_u f = u[f]$$

Similarly, there are no changes in the component equations:

$$df = f_{,\alpha}\omega^\alpha\ \ \text{(expansion of df in aribitrary basis)}$$

$$f_{,\alpha} = \partial_\alpha f = e_\alpha[f]\ \ (\textit{way to calculate components od df})$$

$$f_{,\alpha} = \frac{\partial f}{\partial x^\alpha}\ \ \text{(if } e_\alpha \text{ is a coordinate basis)}$$

Tensors

Tensor S must reside at a specific event P_0 just as any vector or 1-form must. And the algebra of component manipulations is the same:

$$S^\alpha_{\beta\gamma} = S(\omega^\alpha, e_\beta, e_g)$$

$$S = S^\alpha_{\beta\gamma} e_\alpha \otimes \omega^\beta \otimes \omega^\gamma$$

$$S(\sigma, u, v) = S^\alpha_{\beta\gamma}\sigma_\alpha u^\beta v^\gamma$$

Commutators

A vector u_0 given only at one point P_0 suffices to compute the derivative $u_0[f] \equiv \partial_{u_0} f$, which is simply a number.A vector field u provides a vector u(P)-which is a differential operator $\partial_{u(P)}$-at each point P in some region of spacetime. This vector field operates on a fuction f to produce another function $u[f] \equiv \partial_u f$. A second vector field v can to operate on this new fucntion, to produce yet another fucntion

$$v[u[f]] = \partial_v(\partial_u f)$$

Equivalently, does the commutator vanish?

$$[u, v][f] \equiv u[v[f]] - v[u[f]]$$

In general the commutator is nonzero, as one sees from a coordinate-based calculation:

$$[u, v][f] = u^\alpha \frac{\partial}{\partial x^\alpha}\left(v^\beta \frac{\partial f}{\partial x^\beta}\right) - v^\alpha \frac{\partial}{\partial x^\alpha}\left(u^\beta \frac{\partial f}{\partial x^b}\right) = \left[(u^\alpha v^\beta_{,\alpha} - v^\alpha u^\beta_{,\alpha})\frac{\partial}{\partial x^b}\right][f]$$

Commutator of two vector fields is a vector field Commutators find application in the distinction between a coordinate-induced basis $\{e_\alpha\} = \{\partial/\partial x^\alpha\}$, and a noncoordinate basis. $\{e_\alpha(P)\}$ is a coordinate - induced basis if and only if $[e_\alpha, e_\beta] = 0$ for all e_α and e_β. In an noncordinate basis one defines the commutation coefficients $c^\alpha_{\mu\nu}$ by

$$[e_\mu, e_\nu] = c^\alpha_{\mu\nu} e_\alpha$$

The enter into the component formula for the commutator of arbitrary vector fields u and v:

$$[u, v] = (u[v^\beta] - v[u^\beta] + u^\mu v^\nu c^\beta_{\mu\nu})e_\beta$$

Manifolds

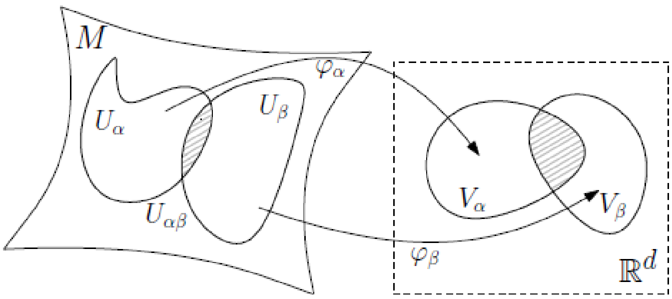
Let as M is a set with a following structure.

- $M = \bigcup\limits_{\alpha \in A} U_\alpha$

- $\forall\, U_\alpha \quad \exists\, (\varphi_\alpha, V_\alpha) : \quad V_\alpha \subset \mathbb{R}^d$ - open set, $\varphi_\alpha : U_\alpha \leftrightarrow V_\alpha$

- $U_{\alpha\beta} = U_\alpha \bigcap U_\beta \neq \varnothing \Rightarrow$ on the $\varphi_\beta(U_{\alpha\beta})$ defines smooth mapping:

$$\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_{\alpha\beta}) \to \varphi_\alpha(U_{\alpha\beta})$$



Than :

def. M is a smooth d-dimentional manifold.

def. U_α is a coordinates environs

def. φ_α is a coordinates mapping.

def. $(U_\alpha, \varphi_\alpha)$ is a map.

def. $x = \varphi_\alpha(P) \in V_\alpha$ is a local coordinates of the point $P \in U_\alpha$

def. $\varphi_\alpha \circ \varphi_\beta^{-1}$ is a weaving mapping.

def. The atlas is the set of all maps of the manifold.

Affine geometry

Geodesics. Free fall is the "natural state of motion"so natural,that the path through spacetime of a freely falling, neutral test body in independent of its structure and composition.

To fix the positions which passes moving body one uses the clock. The clock ticks as the body moves, labeling each event on its trajectory with a number: the timeλ the body was there. Result: the free-fall trajectory is not just a sequence of points; it is a parametrized sequence, a curve $P(\lambda)$.

λ is unique only up to linear transfomations

$$\lambda_{new} = a\lambda_{old} + b$$

"b"is a new origin of clock time and "a"is a ratio of new unts to old.

In the curved spacetime of Einstein these parametrized free-all trajectories are the straightest of all posiible curves.Consequently, one gives these trajectories the same name "geodesics". λ is called "affine parameter".

Parallel transport as defined by geodesics. Transport any suffeciently short stretch of a curve AX parallel to itself along curve AB to point B as follows:

1. Take some point M along AB close to A. Take geodesic XM through X and M. Take any affine parametrization λ of XM and define a unique point N by the condition $\lambda_N = \frac{1}{2}(\lambda_X + \lambda_M)$

2. Take geodesic that starts at A and passes through M, and extend it an equal parameter increment to point P.

3. Curve MP gives vector AX as propagated parallel to itself from A to M. This construction certainly yields parallel transport in flat spacetime.It is local.

4. Repeat over and over, and eventually end up with AX propagated parallel to itself from A to B. Call this construction "Schild's Ladder".

5. Result of propagating AX parallel to itself from A to B depends on choice of world line AB.

Covariant differentiation as defined by parallel transport. Ask how rapidly a vector field v is changing along a curve with tangent vector $u = d/d\lambda$. The Answer, $dv/d\lambda \equiv \nabla_u v \equiv$"rate of change of v along u"is constructed by the following obvious procedure: (1) Take v at $\lambda = \lambda_0 + \varepsilon$. (2) Parallel transportit back to $\lambda = \lambda_0$. (3) Calculate how much it differs from v there. (4) Divide by ε and take limit as $\varepsilon \to 0$:

$$\nabla_u v = \lim_{\varepsilon \to 0} \frac{v(\lambda_0 + \varepsilon) - v(\lambda_0)}{\varepsilon}$$

Geodesics as defined by parallel transport or covariant differentiation. The Shild's ladder construction process for parallel transport, applied to the tangent vector of a geodesic guarantees: a geodesic parallel transports its own tangent vector along itself.

Parallel transport and covariant derivative.
Covariant derivative, basic propeties:
Symmetry: $\nabla_u v - \nabla_v u = [u, v]$ for any vector fields u and v;
Chain rule: $\nabla_u (fv) = f\nabla_u v + v\partial_u f$ for any function f, vector field v, vector u;
Addivity: $\nabla_u (v + w) = \nabla_u v + \nabla_u w$ for any vector fields vand w, vector u;
$\nabla_{au+bn} v = a\nabla_u v + b\nabla_n v$ for any field v, vectors or vector fields u and n, and numbers or functions a and b.
Any rule ∇, for producing new vector fields from old, that satisfies these four conditions, is called by differential geometers a "symmetric covariant derivative". Given the geodesics of spacetime, or of any other manifold, one can construct a unique corresponding covariant derivative by the Schild's ladder procedure. Given

any covariant derivative, one can discuss parallel transport via the equation:

$$dv/d\lambda \equiv \nabla_u v = 0 \Leftrightarrow$$

$$\Leftrightarrow \textit{the vector field vis parallel transported along the vector } u = d/d\lambda$$

and one can test whether any curve is a geodesic via

$$\nabla_u u = 0 \Leftrightarrow \textit{ the curve } P(\lambda) \textit{ with tangent vector } u = d/d\lambda \textit{ parallel}$$

$$\textit{transports its own tangent vector } u \Leftrightarrow \ P(\lambda) \textit{is a geodesic.}$$

Thus a knowledge of all geodesics is completely equivalent to a knowledge of the covariant derivative.

The covariant derivative ∇ generalizes to curved spacetime the flat-space gradient∇.

Parallel transport and covariant derivative: component approach. In curved spacetime each event has its own tangent space, and each tangent space requires a basis of its own. As one travels from event to event, comparing their bases via parallel transport, one sees the bases twist and turn.In no other way can they accommodate themselves to the curvature of spacetime. Bases at points P_0 and P_1, which are the same when compared by parallel transport along one curve, must differ when comapred along another curve.

To quantify the twisting and turning of a field of basis vectors $\{e_\alpha(P)\}$ and forms $\{\omega^\alpha(P)\}$, use the covariant derivative. Examine the changes in vector fields along a basis vector e_β , abbereviating

$$\nabla_{e_\beta} \equiv \nabla_\beta$$

and especially examine the rate of change of same basis vector: $\nabla_b e_\alpha$. This rate of change is itself a vector, so it can be expanded in terms of the basis:

$$\nabla_\beta e_\alpha = e_\mu \Gamma^\mu_{\alpha\beta}$$

and the resultant "connection coefficients" $\Gamma^{\mu}_{\alpha\beta}$ can be calculated by projection on the basis 1-forms:

$$< \omega^{\mu}, \nabla_{\beta} e_{\alpha} >= \Gamma^{\alpha}_{\alpha\beta}$$

Because the basis 1-forms are "locked into"the basis vectors, these same connection coefficients $\Gamma^{\nu}_{\alpha\beta}$ tell how the 1-form basis changes from point to point:

$$\nabla_{\beta}\omega^{\nu} = -\Gamma^{\nu}_{\alpha\beta}\omega^{\alpha}$$

$$< \nabla_{\beta}\omega^{\nu}, e_{\alpha} >= -\Gamma^{\nu}_{\alpha\beta}$$

The connection coeffcients do even more. They allow one to calculate the components of the gradient of an arbitrary tensor S. In a Lorentz frame of flat spacetime, the components of ∇S are obtained by letting the basis vectors $e_{\alpha} = \partial P/\partial x^{\alpha} = \partial/\partial x^{\alpha}$ act on the components of S. Thus for a $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ tensor field S one finds that

$$\nabla S \textit{ has components } S^{\alpha}_{\beta\gamma;\delta} = \frac{\partial}{\partial x^{\delta}}[S^{\alpha}_{\beta\gamma}]$$

Not so in curved spacetime, or even in a non-Lorentz basis in flat spacetime. There the basis vectors turn, twist, expand, and contract, so even if S were constant ($\nabla S = 0$), its components on the twisting basis vectors would vary. The connection coefficients, properly applied, will compensate for this twisting and turning. The components of ∇S, called $S^{\alpha}_{\beta\gamma;\delta}$ so that

$$\nabla S = S^{\alpha}_{\beta\gamma;\delta} e_{\alpha} \otimes \omega^{\beta} \otimes \omega^{\gamma} \otimes^{\delta}$$

can be calculated from those of S by the usual flat-space method, plus a correction applied to each index:

$$S^{\alpha}_{\beta\gamma;\delta} = S^{\alpha}_{\beta\gamma;\delta} + S^{\mu}_{\beta\gamma}\Gamma^{\alpha}_{\mu\delta} - S^{\alpha}_{\beta\mu}\Gamma^{\mu}_{\gamma\delta} - S^{\alpha}_{\mu\gamma}\Gamma^{\mu}_{\gamma\delta}$$

Here

$$S^{\alpha}_{\beta\gamma,\delta} \equiv e_{\delta}[S^{\alpha}_{\beta\gamma}] \equiv \partial_{e_{\delta}} S^{\alpha}_{\beta\gamma}$$

For the components of ∇S, introduces special notation, $DS^{\alpha}_{\beta\gamma}/d\lambda$:

$$\nabla_u S = (DS^{\alpha}_{\beta\gamma,\delta}\beta\gamma/\lambda) e_\alpha \otimes \omega^\beta \otimes \omega^\gamma;$$

$$\frac{DS^{\alpha}_{\beta\gamma}}{d\lambda} = S^{\alpha}_{\beta\gamma;\delta}u^\delta = (S^{\alpha}_{\beta\ g,\delta} + \ correction\ trems)u^\delta$$

Since for any f

$$f_{,\delta}u^\delta = \partial_u f = df/d\lambda$$

this deduces to

$$\frac{DS^{\alpha}_{\beta\gamma}}{d\lambda} = \frac{dS^{\alpha}_{\beta\gamma}}{d\lambda} + S^{\mu}_{\beta\gamma}\Gamma^{\alpha}_{\mu\delta}u^\delta - S^{\alpha}_{\mu\gamma}\Gamma^{\mu}_{\beta\gamma}u^\delta - S^{\alpha}_{\beta\mu}\Gamma^{\alpha}_{\gamma\delta}u^\delta$$

Geodesic equation The geodisics were the curves whose tangent vectors, $u = dP/d\lambda$, satisfy $\nabla_u u = 0$. Let a coordinate system $\{x^\alpha(P)\}$ be given. Let it induce basis vectors $e_\alpha = \partial/\partial x^\alpha$ into the tangent space at each event. Let the coonection coefficients $\Gamma^{\alpha}_{\beta\gamma}$ for this "coordinate basis"be given. Then the component version of the geodesic equation $\nabla_u u = 0$ becomes a differential equation for the geodesic $x^\alpha(\lambda)$:

$$u = \frac{d}{d\lambda} = \frac{dx^\alpha}{d\lambda^\alpha}\frac{\partial}{\partial x^\alpha} \qquad \Longrightarrow \qquad components\ of\ u\ are\ u^\alpha = \frac{dx^\alpha}{d\lambda}$$

$$then\ components\ of\ \nabla_u u = 0\ are$$

$$0 = u^{\alpha}_{;\beta}u^\beta = (u^{\alpha}_{,\beta} + \Gamma^{\alpha}_{\gamma\beta}u^\gamma)u^\beta = \frac{\partial}{\partial x^\beta}\left(\frac{dx^\alpha}{d\lambda}\right)\frac{dx^\beta}{d\lambda} + \Gamma^{\alpha}_{\gamma\beta}\frac{dx^\gamma}{d\lambda}\frac{dx^\beta}{d\lambda}$$

which reduces to the differential equation

$$\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^{\alpha}_{\gamma\beta}\frac{dx^\gamma}{d\lambda}\frac{dx^\beta}{d\lambda} = 0$$

This component version of the geodesic equation gives an analytic method for constructing the parallel transport law from a knowledge of the geodesics.

Geodesic deviation and Riemann curvature

Geodesic deviation and spacetime curvature Focus attention on a family of geodesics. Let one geodesic be distinguished from another by the value of a sector parameter n. THe typical point P on the typecal geodesic will be a continous, doubly differentiable function of the selector parameter n and the affine parameter λ; thus $P = P(\lambda, n)$. The tangent vector $u = \dfrac{\partial}{\partial\lambda}$ and separation vector $n = \dfrac{\partial}{\partial n}$ "Geodesic separation" n is displacement from point on fiducal geodesic to point on nearby geodesic characterized by same value of affine parameter λ. Geodesic separation changes with respect to λ at a rate given by the equation of geodesic deviation

$$\nabla_u \nabla_u n + R(\ldots, u, n, u) = 0$$

It terms of components of the Riemann tensor the driving force is

$$R(\ldots, u, n, u) = e_\alpha R^\alpha_{\beta\gamma\delta} u^\beta n^\gamma u^\delta$$

The components of the Riemann curvature tensor in a coordinate frame are given in terms of the connection coefficients by the formula

$$R^\nu_{\gamma\beta\alpha} = \Gamma^\nu_{\beta\gamma,\alpha} - \Gamma^\nu_{\alpha\gamma,\beta} + \Gamma^\lambda_{\beta\gamma}\Gamma^\nu_{\alpha\lambda} - \Gamma^\lambda_{\alpha\gamma}\Gamma^\nu_{\beta\gamma}$$

How to derive this formula? Consider the action of the commutator $[\nabla_u, \nabla_v]$ on the vector a then :

$$\left[\begin{array}{l} \nabla_u v = \nabla_u (v^\alpha e_\alpha) = u^\beta v^\alpha_{;\beta} e_\alpha + v^\alpha u^\beta \omega^\alpha_\beta (e_\gamma) e_\alpha \\ u[v] = u^\beta \dfrac{\partial v}{\partial x^\beta}\ ;\ u[e_\alpha] = \omega^\alpha_\mu (u) = \omega^\alpha_\mu (u^\gamma e_\alpha) = u^\alpha \omega^\alpha_\mu (e_\gamma) e_\alpha \end{array}\right]$$

$$[\nabla_u, \nabla_v] a = \nabla_u (v^\alpha a^\nu_{;\alpha} e_\nu) - \nabla_v (u^\alpha a^\nu_{;\alpha} e_\nu) = u^\beta (v^\alpha a^\nu_{;\alpha} e_\nu)_{;\beta} - v^\alpha (u^\beta a^\nu_{;\beta} e_\nu)_{;\alpha} =$$

$$= e_\nu(u^\beta v^\alpha_{;\beta} a^\nu_{;\alpha} + u^\beta v^\alpha a^\nu_{;\alpha;\beta} - v^\alpha u^\beta_{;\alpha} a^\nu_{;\beta} - v^\alpha u^\beta a^\nu_{;\beta;\alpha}) =$$

$$= e_\nu(u^\beta v^\alpha (a^\nu_{;\alpha;\beta} - a^\nu_{;\beta;\alpha}) + a^\nu_{;\alpha}(u^\beta v^\alpha_{;\beta} - v^\alpha u^\beta_{;\alpha})) = \nabla_{[u,v]} a \quad + \quad \underbrace{R^\nu_\mu(u,v) a^\mu e_\nu}_{(\nabla_u \nabla_v - \nabla_v \nabla_u - \nabla_{[u,v]})a}$$

$$\left[\nabla_u \nabla_v - \nabla_v \nabla_u - \nabla_{[u,v]}\right] a = u^\alpha u^\beta a^\gamma_{[;\alpha;\beta]} e_\gamma = R(\cdot, a, u, v) = R^\nu_\mu(u,v) a^\mu e_\nu$$

$$\begin{bmatrix} a^\nu_{;\alpha}(u^\beta v^\alpha_{;b} - v^\alpha u^\beta_{;\alpha}) = e_\gamma [u,v] a^\gamma_{;\alpha} = \nabla_{[u,v]} a \\ R^\mu_\nu = (a^\gamma_{;\alpha;\beta} - a^\gamma_{;\beta;\alpha}) \\ (a^\nu_{;\alpha;\beta} - a^\nu_{;\beta;\alpha}) = R^\gamma_{\mu\alpha\beta} a^\mu \to \frac{1}{2} R^\mu_{\nu\alpha\beta}\, \omega^\alpha \wedge \omega^\beta \\ a^\nu_{;\nu} = a^\mu_{,\nu} + \Gamma^\mu_{\nu\alpha} a^\alpha \end{bmatrix}$$

$$a^\nu_{;\beta;\alpha} - a^\nu_{;\alpha;\beta} = R^\nu_{\gamma\beta\alpha} a^\gamma = (a^\nu_{,b} + \Gamma^\nu_{\beta\gamma} a^\gamma)_{;\alpha} - (a^\nu_{,\alpha} + \Gamma^\nu_{\alpha\gamma} a^\gamma)_{;\beta} =$$

$$= (\Gamma^\nu_{\beta\gamma} a^\gamma)_{,\alpha} - (\Gamma^\nu_{\beta\gamma})_{,\beta} + \Gamma^\nu_{\alpha\lambda}(a^\lambda_{,\beta} + \Gamma^\lambda_{\beta\gamma} a^\gamma) - \Gamma^\nu_{\beta\lambda}(a^\lambda_{,\alpha} + \Gamma^\lambda_{\alpha\gamma} a^\gamma) =$$

$$= (\Gamma^\nu_{\beta\gamma,\alpha} - \Gamma^\nu_{\alpha\gamma,\beta}) a^\gamma + (\Gamma^\nu_{\alpha\lambda} \Gamma^\lambda_{\beta\gamma} - \Gamma^\nu_{\beta\gamma} \Gamma^\lambda_{\alpha\gamma}) a^\gamma$$

$$\Longrightarrow R^\nu_{\gamma\beta\alpha} = \Gamma^\nu_{\beta\gamma,\alpha} - \Gamma^\nu_{\alpha\gamma,\beta} + \Gamma^\lambda_{\beta\gamma} \Gamma^\nu_{\alpha\lambda} - \Gamma^\lambda_{\alpha\gamma} \Gamma^\nu_{\beta\gamma}$$

Flatness is equivalent to zero Riemann curvature To say that space or spacetime or any other manifold is flat is to say that there exist a coordinate system $\{x^\alpha(P)\}$ in which all geodesics appear straight:

$$x^\alpha(\lambda) = a^\alpha + b^\alpha \lambda$$

They can appear so if and only if the connection coefficients in the geodesic equation

$$\frac{d^2 x^\beta}{d\lambda^2} + \Gamma^\beta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0$$

expressed in the same coordinate system, all vanish: $\Gamma^\beta_{\mu\nu} = 0 \implies R^\beta_{\gamma\mu\nu} = 0$

Curvature] of the Schwarzschild metric

Below I tried to deduce a Einstein's equation within Schwarzschild's metric.

$ds^2 = e^{2\phi}dt^2 - e^{2\psi}dr^2 - r^2 d\theta^2 - r\sin d\varphi^2$

$\omega^0 = e^{\phi}dt$	$dt = \omega^0 e^{-\phi}$	$d(dx^\mu) = 0$
$\omega^1 = e^{\psi}dr$	$dr = \omega^1 e^{-\psi}$	$dx^\mu \wedge dx^\mu = 0$
$\omega^2 = r d\theta$	$d\theta = \omega^2 r^{-1}$	$d\omega^\mu = -\omega^\mu_\alpha \wedge \omega^\alpha$
$\omega^3 = r\sin d\varphi$	$d\varphi = \omega^3 (r\sin\theta)^{-1}$	$d(f\alpha) = df \wedge \alpha + f d\alpha$
		$df = f_{,\alpha} dx^\alpha$

$$d\omega^0 = \phi' e^{\phi} dr \wedge dt = \phi' e^{-\psi}\omega^1 \wedge \omega^0 = -\omega^0_\alpha \wedge \omega^\alpha$$

$$d\omega^1 = \psi' e^{\psi}(dr \wedge dr) = 0$$

$$d\omega^2 = dr \wedge \theta = \tfrac{1}{r} e^{-\psi}\omega^1 \wedge \omega^2$$

$$d\omega^3 = \sin\theta(dr \wedge \varphi) + r\cos\theta(d\theta \wedge d\varphi) = \tfrac{1}{r} e^{-\psi}\omega^1 \wedge \omega^3 + \tfrac{ctg\theta}{r}\,\omega^2 \wedge \omega^3$$

$$\left[\begin{array}{l} \eta_{\mu\nu} = -\omega_{\mu\nu} + \omega_{\nu\mu} = 0 \\ \omega_{\mu\nu} = -\omega_{\nu\mu} \\ \omega_{00} = 0 \\ \omega_{10} = -\omega_{01} \\ \omega^1_0 = \omega^0_1 \\ \omega^\alpha_0 = \omega^0_\alpha \end{array}\right]$$

$$\begin{cases} d\omega^0 = -\omega^0_1 \wedge \omega^1 \\ d\omega^1 = 0 = -\omega^1_0 \wedge \omega^0 \\ d\omega^2 = -\omega^2_1 \wedge \omega^1 = -\frac{e^{-\psi}}{r}\,\omega^2 \wedge \omega^1 \\ d\omega^3 = -\omega^3_1 \wedge \omega^1 - \omega^3_2 \wedge \omega^2 = -\frac{-\psi}{r}\,\omega^3 \wedge \omega^1 - \frac{ctg\theta}{r}\,\omega^3 \wedge \omega^2 \\ \omega^0_1 = \omega^1_0 = \phi' e^{-\psi} \\ \omega^0_2 = \omega^2_0 = \omega^0_3 = \omega^3_0 = 0 \\ \omega^1_2 = -\omega^2_1 = -\frac{e^{-\psi}}{r}\omega^2 \\ \omega^1_3 = -\omega^3_1 = - - \frac{e^{-\psi}}{r}\omega^3 \end{cases}$$

$R^{\mu}_{\nu} = d\omega^{\mu}_{\nu} + \omega^{\mu}_{\lambda} \wedge \omega^{\lambda}_{\nu}$

$R^0_1 = d\omega^0_1 + \omega^0_{\lambda} \wedge \omega^{\lambda}_1 = \big[d\omega^0_1 = d(\phi' e^{-\psi} \omega^0) = (\phi' e^{-\psi})' e^{-\psi} \omega^1 \wedge \omega^0 +$

$+ (\phi' e^{-\psi})^2 \omega^1 \wedge \omega^0 \big] = (\phi' e^{-\psi})' e^{-\psi} \omega^1 \wedge \omega^0 + (\phi' e^{-\psi})^2 \omega^1 \wedge \omega^0$

$R^0_2 = R^2_0 = d\omega^0_2 + \omega^0_{\lambda} \wedge \omega^{\lambda}_2 = \omega^0_1 \wedge \omega^1_2 = -\phi' e^{-\psi} \ \frac{e^{-\psi}}{r} \ \omega^0 \wedge \omega^2 = -\phi' \frac{e^{-2\psi}}{r} \ \omega^0 \wedge \omega^2$

$R^0_3 = R^3_0 = d\omega^0_3 + \omega^0_{\lambda} \wedge \omega^{\lambda}_3 = \omega^0_1 \wedge \omega^1_3 = -\phi' e^{-\psi} \ \frac{e^{-\psi}}{r} \ \omega^0 \wedge \omega^3 = -\phi' \frac{e^{-2\psi}}{r} \ \omega^0 \wedge \omega^3$

$R^1_2 = -R^2_1 = d\omega^2_1 + \omega^2_{\lambda} \wedge \omega^{\lambda}_1 = (-\frac{e^{-\psi}}{r})' e^{-\psi} \omega^1 \wedge \omega^2 - (\frac{e^{-\psi}}{r})^2 \ \omega^1 \wedge \omega^2$

$R^1_3 = -R^3_1 = d\omega^3_1 + \omega^3_{\lambda} \wedge \omega^{\lambda}_1 = -(\frac{e^{-\psi}}{r})' \ e^{-\psi} \omega^1 \wedge \omega^3 - (-\frac{e^{-\psi}}{r})^2 \ \omega^1 \wedge \omega^3 +$

$+ \frac{e^{-\psi}}{r^2} \ ctg\theta \ \omega^2 \wedge \omega^3 + \frac{ctg\theta}{r} \ \omega^3 \wedge \omega^2 \ \frac{e^{-\psi}}{r} = -(\frac{e^{-\psi}}{r})' \ e^{-\psi} \omega^1 \wedge \omega^3 - (\frac{e^{-\psi}}{r})^2 \ \omega^1 \wedge \omega^3$

$R^2_3 = -R^3_2 = d\omega^3_2 + \omega^3_{\lambda} \wedge \omega^{\lambda}_2 = \Big[d\omega^3_2 = d(\frac{ctg\theta}{r} \ \omega^3) + \omega^3_1 \wedge \omega^1_2;$

$d(\frac{ctg\theta}{r} \ \omega^3) = \frac{ctg\theta}{r^2} \ e^{-\psi} \ \omega^1 \wedge \omega^3 + \frac{1}{r^2 \sin^2\theta} \ \omega^2 \wedge \omega^3 \ +$

$+ (-\frac{ctg\theta}{r}) \ (\frac{e^{-\psi}}{r} \ \omega^1 \wedge \omega^3 + \frac{crtg\theta}{r} \ \omega^2 \wedge \omega^3) \Big] = \frac{1}{r^2} \ \omega^2 \wedge \omega^3 +$

$+ (-\frac{e^{-\psi}}{r} \ \omega^3 \wedge \omega^2) = (\frac{1}{r^2} + \frac{e^{-\psi}}{r}) \ \omega^2 \wedge \omega^3$

$R^{\mu}_{\nu} = \frac{1}{2} R^{\mu}_{\nu\alpha\beta} \ \omega^{\alpha} \wedge \omega^{\beta}$

$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}$

$R_{00} = R^1_{010} + R^2_{020} + R^3_{030} = (\phi' e^{-\psi})' \ e^{-\psi} + (\phi' e^{-\psi})^2 - 2\phi' \ \frac{e^{-2\psi}}{r}$

$R^1_{010} = R^0_{110} = \big((\phi' e^{-\psi})' e^{-\psi} + (\phi' e^{-\psi})^2 \big) \ ;$

$R^2_{020} = R^0_{220} = -\phi' \frac{e^{-2\psi}}{r};$

$R^3_{030} = R^0_{330} = -\phi' \frac{e^{-2\psi}}{r};$

non diagonal elements are equiv 0

$R_{11} = R^0_{101} + R^1_{111} + R^2_{121} + R^3_{131} = -(\phi' e^{-\psi})' e^{-\psi} - (\phi' e^{-\psi})^2 + (\frac{e^{-\psi}}{r})' e^{-\psi} +$

$+ (\frac{e^{-\psi}}{r})^2 + (\frac{e^{-\psi}}{r})' e^{-\psi} + (e^{-\psi})^2 = -(\phi' e^{-\psi})' e^{-\psi} - (\phi' e^{-\psi})^2 - \frac{2e^{-2\psi}}{r} \psi'$

$R_{22} = R^0_{202} + R^1_{212} + R^2_{222} + R^3_{232} = -\phi' \frac{e^{-2\psi}}{r} \ + (-\frac{e^{-\psi}}{r})' e^{-\psi} - (e^{-\psi})^2 - \frac{1}{r^2} - \frac{e^{-\psi}}{r} =$

$= -\phi' \frac{e^{-2\psi}}{r} + \frac{e^{-2\psi}}{r} \ \psi' + \frac{e^{-2\psi}}{r^2} - \frac{e^{-2\psi}}{r^2} - \frac{1}{r^2} - \frac{e^{-\psi}}{r} = -\phi' \frac{e^{-2\psi}}{r} + \frac{e^{-2\psi}}{r} \ \psi' - \frac{1}{r^2} - \frac{e^{-\psi}}{r}$

$R_{33} = R^0_{303} + R^1_{313} + R^2_{323} + R^3_{333} = \frac{e^{-\psi}}{r} \psi' + \frac{1}{r^2} + \frac{e^{-\psi}}{r} - \phi' \frac{e^{-2\psi}}{r}$

Solution of geodesic equations

Several solutions for geodesic equations within Schwarzschild's metric have been received and several possible orbits of planet movement have been built through Maple's Geodesic package. Not all possible orbits have been received, since some of solutins, which have been received in Maple, were bulky and their analysis is quiet difficult and I have not managed to implement them.

General form equasion:

$$\frac{d^2}{d^2\tau}X^\mu(\tau)+\Gamma^\mu_{\alpha\nu}\left(\frac{d}{d\tau}X^\nu(\tau)\right)\left(\frac{d}{d\tau}X^\alpha(\tau)\right)$$

The straight-lines solution computed directly

$$G := \left[\frac{d^2}{d\tau^2}\phi(\tau) = -\frac{2\left(\frac{d}{d\tau}\phi(\tau)\right)\left(\cos(\theta(\tau))\,r(\tau)\left(\frac{d}{d\tau}\theta(\tau)\right)+\sin(\theta(\tau))\left(\frac{d}{d\tau}r(\tau)\right)\right)}{r(\tau)\sin(\theta(\tau))},\right.$$

$$\frac{d^2}{d\tau^2}\theta(\tau) = \frac{\sin(\theta(\tau))\cos(\theta(\tau))\left(\frac{d}{d\tau}\phi(\tau)\right)^2 r(\tau)-2\left(\frac{d}{d\tau}r(\tau)\right)\left(\frac{d}{d\tau}\theta(\tau)\right)}{r(\tau)},$$

$$\frac{d^2}{d\tau^2}r(\tau) = \frac{1}{(-r(\tau)+2m)\,r(\tau)^3}\left(4\left(-\frac{r(\tau)}{2}+m\right)^2(\cos(\theta(\tau))\right.$$

$$+1)\,r(\tau)^3(\cos(\theta(\tau))-1)\left(\frac{d}{d\tau}\phi(\tau)\right)^2-4\left(-\frac{r(\tau)}{2}+m\right)^2 r(\tau)^3\left(\frac{d}{d\tau}\theta(\tau)\right)^2$$

$$\left.+4\left(-\frac{r(\tau)}{2}+m\right)^2 m\left(\frac{d}{d\tau}t(\tau)\right)^2-m\left(\frac{d}{d\tau}r(\tau)\right)^2 r(\tau)^2\right),$$

$$\left.\frac{d^2}{d\tau^2}t(\tau) = -\frac{2m\left(\frac{d}{d\tau}r(\tau)\right)\left(\frac{d}{d\tau}t(\tau)\right)}{r(\tau)\,(r(\tau)-2m)}\right]$$

This system of ODEs, as is, it is out of reach of the DE solvers of the system mainly due to the presence of non-rational objects like sin and cos having for arguments one of the unknowns of the system, $\theta(\tau)$. On the other hand, we know the geodesics for the Schwarzschild metric describe the motion of particles of infinitesimal mass in the gravitational field of a central fixed large mass. So to investigate the solvability of these equations one can assume $\theta(\tau)$ is constant an due to the rotational symmetry choose a value for it that simplifies the equations, for example $\theta(\tau) = \dfrac{\pi}{2}$

Then new equations form:

$$Y := \left[\frac{d^2}{d\tau^2}\phi(\tau) = -\frac{2\left(\frac{d}{d\tau}\phi(\tau)\right)\left(\frac{d}{d\tau}r(\tau)\right)}{r(\tau)}, \right.$$

$$0 = 0,$$

$$\frac{d^2}{d\tau^2}r(\tau) = \frac{-4\left(-\frac{r(\tau)}{2}+m\right)^2 r(\tau)^3\left(\frac{d}{d\tau}\phi(\tau)\right)^2}{(-r(\tau)+2m)\,r(\tau)^3} +$$

$$\frac{+4\left(-\frac{r(\tau)}{2}+m\right)^2 m\left(\frac{d}{d\tau}t(\tau)\right)^2 - m\left(\frac{d}{d\tau}r(\tau)\right)^2 r(\tau)^2}{(-r(\tau)+2m)\,r(\tau)^3},$$

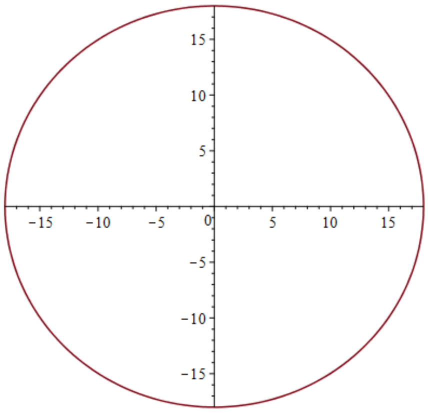
$$\left. \frac{d^2}{d\tau^2}t(\tau) = -\frac{2m\left(\frac{d}{d\tau}r(\tau)\right)\left(\frac{d}{d\tau}t(\tau)\right)}{r(\tau)\,(r(\tau)-2m)} \right]$$

The first solution:

$$\begin{cases} r(\tau) = 6m \\ \phi(\tau) = C2\tau + C3 \\ t(\tau) = -6\sqrt{6}\, m\, \phi(\tau) + C1 \text{ or } t(\tau) = 6\sqrt{6}\, m\, \phi(\tau) + C1 \end{cases}$$

It is a circular orbit

Let $m = 3$, $C1 = 0$, $C2 = 1$, $C3 = 1$, $\tau = t$

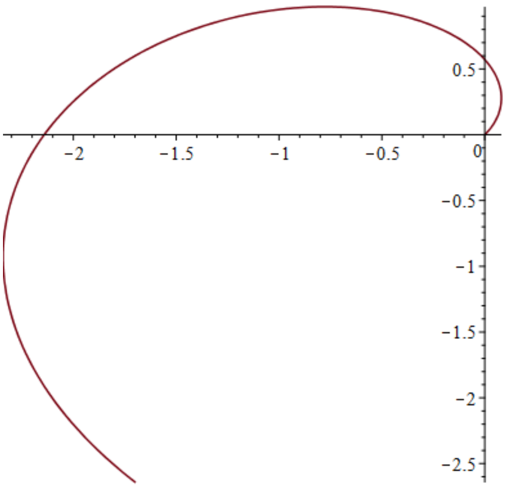


The second solution:

$$\begin{cases} r(\tau) = C4 \\ \phi(\tau) = C2\tau + C3 \\ t(\tau) = \int \frac{\sqrt{r(\tau)\, m}\, r(\tau) \left(\frac{d}{d\tau}\right)}{m} d\tau + C1 \text{ or } t(\tau) = -\int \frac{\sqrt{r(\tau)\, m}\, r(\tau) \left(\frac{d}{d\tau}\right)}{m} d\tau + C1 \end{cases}$$

The body falls down in the center spirally

Let $C4 = -t$, $C2 = -1$, $C3 = 1$, $C1 = 0$, $\tau = t$

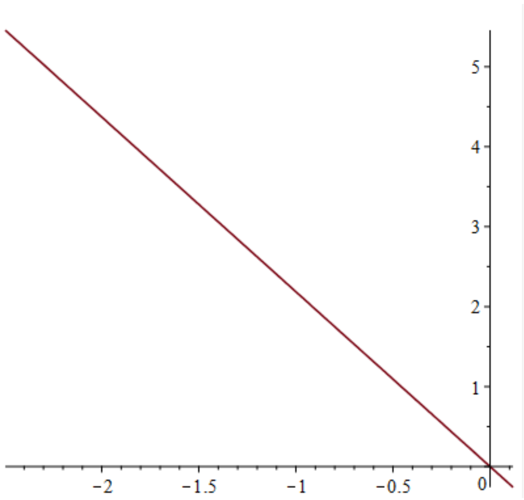


The third solution:

$$\begin{cases} r(\tau) = C3\tau + C4 \\ \phi(\tau) = C2 \\ t(\tau) = -r(\tau) - 2m\ln(r(\tau) - 2m) + C1 \end{cases}$$

Just falls down in the center

Let $C1 = 0$, $C2 = t$, $C3 = -1$, $C4 = 6$, $\tau = t$



Conclusion. During the work I got acquainted with huge amount of information. At first, I have greatly improved my skills of working with mathematical objects. I have learned a lot of gravity and behavior of objects in gravitational field. I have managed to deduce Riman's tensor and curvature of Schwarzschild metric, also the numerical solution of geodesic equations within Schwarzschild metric has been found and several Kepler's orbits have been built. Exept that, my scientific director gave me knowledges of other branches of physics, such as quantum physics, static physics and atomic physics. By ending my report, I wanted to express gratitude to Hovik Grigorayn and Alexander Ayriyan, to all LIT staff and JINR managers for care during the work.

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