JOINT INSTITUTE FOR NUCLEAR RESEARCH

Bogolyubov Laboratory of Theoretical Physics

FINAL REPORT ON THE START PROGRAMME

Gravitating Skyrmions with gauge coupling

Supervisor:
Dr. Yakov Mikhailovich Shnir

Student:
Roman Kirichenkov, Belarus
Belarusian State University

Participation period:
August 13 - September 23
Summer Session 2023

Dubna, 2023

Contents

Abstract

The Skyrme theory is one the nonlinear theories that has topological invariant due to its mapping structure. Nevertheless, there were made no or little attempts to analyze the behavior of solutions in a highly curved space-time while having a gauge coupling. This situation can appear in a vast regions in the space. Combined gravitational and electromagnetic interactions can lead to the nontrivial results that's we are to study.

Main part

1. Introduction

The Skyrme field is a field that can be concerned as an effective model of strong interactions in the low energy area. Despite the theory of Skyrmion field still requires additional analysis and investigations it has already achieved significant correspondence to observational data.

Having a great potential in describing the nuclear structure, the model possesses several non-linear terms that bring the whole interest to the theory. This work can be considered as a brief review of the basic Skyrme theory with gravitational and electromagnetic interactions. It is worth mentioning that this scalar field appears to be not only the effective model of strong interactions but can play important role on the cosmological scales.

Firstly we will consider Skyrmions with gravitational interaction only. Then we will move to the Skyrme field that besides gravitational interaction has an electromagnetic one.

But it is important to begin with the basic concepts of the Skyrme theory, so that's why we will begin with a short discussion of the origins of this work.

2. Skyrmion

1. Basic Field

Let's begin our consideration from the basic variant of the field that arises in the Skyrme theory. The basis of the theory is a field U, the $SU(2)$-scalar. This means the field U is characterized by complex 2×2 matrix which belongs to the matrix representation of the $SU(2)$ group.

The fact that matrix U belongs to the matrix representation of the $SU(2)$ group can be described by following expressions:

$$\det U = 1, \quad UU^\dagger = \mathbb{I}$$

where † stands for Hermite conjugate, $\mathbb{I}$ – identity matrix.

It follows from the conditions above that matrix U has to have the form:

$$U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \quad \text{and} \quad |a| + |b| = 1, \quad a, b \in \mathbb{C}$$

If we express the complex numbers a and b in the algebraic form:

$$a = \sigma + i\varphi_3, \quad b = \varphi_2 + \varphi_1$$

then the condition $U \in SU(2)$ takes the form:

$$U = \begin{pmatrix} \sigma + i\varphi_3 & \varphi_2 + i\varphi_1 \\ -\varphi_2 + i\varphi_1 & \sigma - i\varphi_3 \end{pmatrix} \quad \text{and} \quad \sigma^2 + \varphi_1^2 + \varphi_2^2 + \varphi_3^2 = 1,$$

where $\sigma, \varphi_1, \varphi_2, \varphi_3 \in \mathbb{R}$

One can simplify the representation of the field U in terms of real fields $\sigma, \varphi_1, \varphi_2, \varphi_3$ by introducing the Pauli matrices:

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

After introducing vectors $\vec{\tau} = (\tau_1, \tau_2, \tau_3)$ and $\vec{\varphi} = (\varphi_1, \varphi_2, \varphi_3)$, the matrix field U then can be represented in the following way:

$$U = \sigma\mathbb{I} + i\vec{\tau} \cdot \vec{\varphi}, \quad \sigma^2 + \vec{\varphi} \cdot \vec{\varphi} = 1$$

2. Basic Lagrangian

Using the filed U, the Lagrangian of the Skyrme theory in its simplest form is:

$$\mathcal{L}_S = \frac{f^2}{16}\mathrm{Sp}(\partial_\mu U \partial^\mu U^\dagger) + \frac{1}{32e^2}\mathrm{Sp}\left(\left[\partial_\mu U U^\dagger, \partial_\nu U U^\dagger\right]\left[\partial^\mu U U^\dagger, \partial^\nu U U^\dagger\right]\right)$$

where parameters f and e are selected for better consistency with experimental data.

This Lagrangian can be simplified more if we introduce the quantity:

$$R_\mu = \partial_\mu U U^\dagger$$

By using this value and the condition $U \in SU(2)$ the Lagrangian takes the form:

$$\mathcal{L}_S = \frac{f^2}{16}\mathrm{Sp}\left(R_\mu R^\mu\right) + \frac{1}{32e^2}\mathrm{Sp}\left(\left[R_\mu, R_\nu\right]\left[R^\mu, R^\nu\right]\right)$$

The Lagrangian $\mathcal{L}_S$ can be rewritten [1] in terms of 4-vector of potentials $\varphi_\mu = (\vec{\varphi}, \sigma)$ – in that case after rescaling $f \to 2f$ and demanding $fe = 1$ the Lagranginan will take the following view:

$$\mathcal{L}_S = \frac{f^2}{2}\left\{\partial_\mu \varphi_\nu \partial^\mu \varphi^\nu - \frac{1}{2}\left(\partial_\mu \varphi_\nu \partial^\mu \varphi^\nu\right)^2 + \frac{1}{2}\left(\partial_\mu \varphi_\lambda \partial_\nu \varphi^\lambda\right)\left(\partial^\mu \varphi_\rho \partial^\nu \varphi^\rho\right)\right\}$$

3. Gravitating Skyrmion

1. Metrics

Now we turn to the consideration of the Skyrmion in the gravitational field. To begin with, we will consider the spherically symmetric case - for this we will introduce the Schwarzschild metric into consideration:

$$g_{\mu\nu} = \mathrm{diag}\left(\left(-\frac{1}{1-\frac{2\kappa m(r)}{r}}\right), -r^2, -r^2\sin^2\theta, \left(1-\frac{2\kappa m(r)}{r}\right)\sigma^2(r)\right)$$

where κ is the coupling constant of gravity and the Skyrm field. This relationship is expressed as follows:

$$æ = \frac{4\pi G f^2}{c^2}$$

Thus, this coupling constant includes both the gravitational parameter G and the parameter f responsible for the Skyrme field. A non-zero value of the coupling constant κ can be interpreted either as a modified value of G at a given value of f or as a modified value of f at a constant value of G, which is physically more correct.

2. Lagrangian and ansatz

The Lagrangian for the gravitational field together with the Skyrme field has the form:

$$\mathcal{L} = \left(-\frac{1}{2}R + \kappa\left\{-\frac{1}{2}\mathrm{Sp}\left(R_\mu R^\mu\right) + \frac{1}{16}\mathrm{Sp}\left(\left[R_\mu, R_\nu\right]\left[R^\mu, R^\nu\right]\right)\right\}\right)\sqrt{-g}$$

To find a spherically symmetric solution, we will look for it in the form:

$$U = \cos\chi(r)\mathbb{I} + i\sin\chi(r)\vec{\tau}\cdot\vec{n}$$

where $\vec{n} = (\sin\theta\cos\psi, \sin\theta\sin\psi, \cos\theta)$ – unit vector normal to the surface of the unit 2-sphere.

Using this ansatz, instead of 4 potentials σ and $\vec{\varphi}$, we will look for a solution for one profile function $\chi(r)$, which defines all 4 potentials simultaneously.

Potentials σ and $\vec{\varphi}$ defined using $\chi(r)$ as follows:

$$\begin{cases} \varphi_1 = \sin\chi(r)\sin\theta\cos\psi \\ \varphi_2 = \sin\chi(r)\sin\theta\sin\psi \\ \varphi_3 = \sin\chi(r)\cos\psi \\ \sigma = \cos\chi(r) \end{cases}$$

3. Topological charge

One of the characteristic features of Skirm's theory is the presence of a conserved quantity that is not a Noether invariant [2]. This conserved quantity appears due to the mapping $S^3 \to S^3$, where domain represents the potentials φ_μ lying on the unit 3-sphere, and target space is our physical space $\mathbb{M}^4$ with an identified point at infinity.

The topological charge in this case is the degree of mapping $S^3 \to S^3$. In the case of using the ansatz specified above, the expression for calculating this charge can be written as:

$$Q = -\frac{1}{\pi}\left(\chi(r) - \frac{\sin 2\chi(r)}{2}\right)\Bigg|_0^{+\infty}$$

At infinity, the profile function is assumed to be equal to zero. Therefore, in order for the topological charge to take a value equal to 1, the condition must be met:

$$\chi(0) = \pi$$

4. Field equations

The field equations in this case are the Euler-Lagrange equations:

$$\frac{\partial}{\partial x_\mu}\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial A_\nu}{\partial x_\mu}\right)} - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0$$

For the case of spherical symmetry and statics, only r remains an independent variable. Variation of the Lagrangian is carried out by the functions $\sigma(r), \chi(r), m(r)$.

Thus, we have three field equations:

$$\begin{cases} \dfrac{d}{dr}\dfrac{\partial \mathcal{L}}{\partial \sigma'} - \dfrac{\partial \mathcal{L}}{\partial \sigma} = 0 \\ \dfrac{d}{dr}\dfrac{\partial \mathcal{L}}{\partial m'} - \dfrac{\partial \mathcal{L}}{\partial m} = 0 \\ \dfrac{d}{dr}\dfrac{\partial \mathcal{L}}{\partial \chi'} - \dfrac{\partial \mathcal{L}}{\partial \chi} = 0 \end{cases}$$

Differentiating the Lagrangian and simplifying the expressions obtained, we obtain three field equations [3]:

$$\begin{cases} m' = \chi'^2 \left(\dfrac{r^2}{2} + \sin^2 \chi \right) N + \sin^2 \chi \left(1 + \dfrac{\sin^2 \chi}{2r^2} \right) \\ \sigma' = \kappa \, \chi'^2 \left(r + \dfrac{2 \sin^2 \chi}{r} \right) \\ \left(\sigma N \left(r^2 + 2 \sin^2 \chi \right) \chi' \right)' = \sigma \left(1 + N \chi'^2 + \dfrac{\sin^2 \chi}{r^2} \right) \sin 2\chi \end{cases}$$

where the function is introduced for brevity $N = 1 - \dfrac{2\kappa m(r)}{r}$

The boundary conditions for this problem will be:

$$\chi(0) = \pi, \chi(\infty) = 0, m(0) = 0, \sigma(\infty) = 1$$

5. Numerical results

The numerical solution of these equations together with the specified boundary conditions were obtained using the package FIDISOL/CADSOL, implemented in Fortran. We will solve these equations while changing the parameter κ.

Since there will be more than one solution for a non-zero value of κ, to analyze the spectrum of solutions, we will plot $\sigma(0)$ from κ and $M = m(\infty)$ from κ.

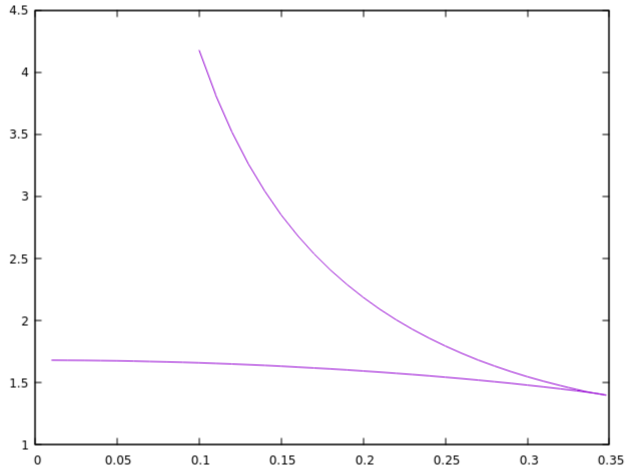

Figure 1: Mass of the system M from the coupling constant κ

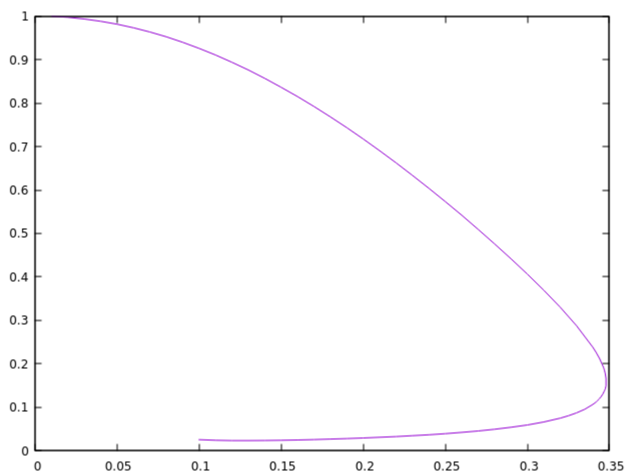

Figure 2: $\sigma(0)$ (or $F_0(0)^*$) from the coupling constant κ

Note 1: Figures 1 and 2 are build for the Lagrangian with additional term $m^2\,\mathrm{Sp}(U-\mathbb{I}), m=1$ and with different type of metric parametrization that has the form:

$$g_{\mu\nu}=\mathrm{diag}(F_1(r,\theta), r^2F_1(r,\theta), r^2\sin^2\theta F_1(r,\theta), -F_0(r,\theta))$$

But this change of the metrics and Lagrangian has no effect on the type of dependence of $\sigma(0)$ or $F_0(0)$ from κ.

From the figures above we can point out that:

1. There is a critical value of $\kappa_{\mathrm{crit}}\approx 0.35$ and after that value there exists no stable solution of the field equations at all.

2. In the range $(0,\kappa_{\mathrm{crit}})$ there exist two separate solutions that merge at κ_{crit} and diverge at $\kappa=0$.

4. Gravitating Skyrmions with charge

1. Changes

Using gravitating Skyrmions as a start point now we can introduce the electromagnetic interaction by doing the following:

1. Changing ordinary differentiation to the covariant one:

$$D_\mu U=\partial_\mu U+igA_\mu\left[Q,U\right],\quad Q\equiv\frac{1}{2}\left(\frac{1}{3}\mathbb{I}+\tau_3\right)$$

   where g is rescaled gauge coupling constant

2. Introducing the electromagnetic contribution to the Lagrangian:

$$\mathcal{L}\rightarrow\mathcal{L}-\frac{1}{2}F_{\mu\nu}F^{\mu\nu},\quad F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$$

3. Introducing mass term $m^2\mathrm{Sp}(U - \mathbb{I})$ officially to stabilize the solutions

4. Replacing spherical metrics to the axial one because the symmetry of the system changes after we introduce the electromagnetic interaction. The appropriate metric will be:

$$g_{11} = F_1(r,\theta), \quad g_{22} = F_1(r,\theta)r^2, \quad g_{33} = F_2(r,\theta)r^2\sin^2\theta$$
$$g_{34} = g_{43} = -F_2(r,\theta)r^2\sin^2\theta\frac{W(r,\theta)}{r}$$
$$g_{44} = -F_0(r,\theta) + F_2(r,\theta)r^2\sin^2\theta\left(\frac{W(r,\theta)}{r}\right)^2$$

2. Numerical results

After the changes are made we can have a look on the same invariants that are obtained by the same solver. We will measure the gauge contribution by changing the value of $\omega = gA_0(\infty)$. $\omega \to 0$ means the eliminating the gauge coupling and returning to the gravitating Skyrmion discussed above.

As expected for such little value of gauge coupling constant g the pictures above are almost the same as for the gravitating Skyrmions without the charge. But it is required to get and analyze the second branch of the solutions that exists for the same reason as for the simple gravitating Skyrmions.

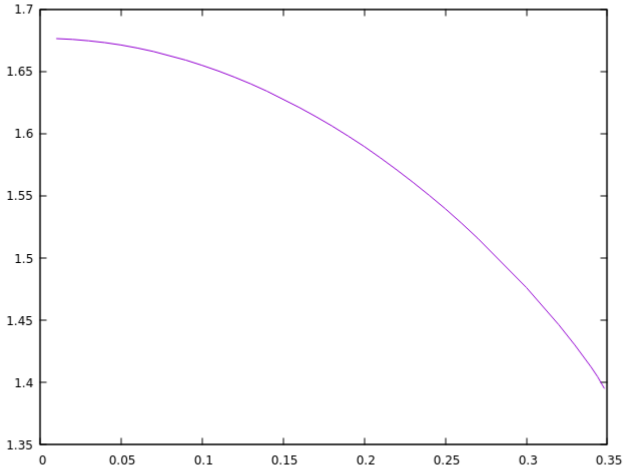

Figure 3: M from the coupling constant κ for $\omega = 0.1$

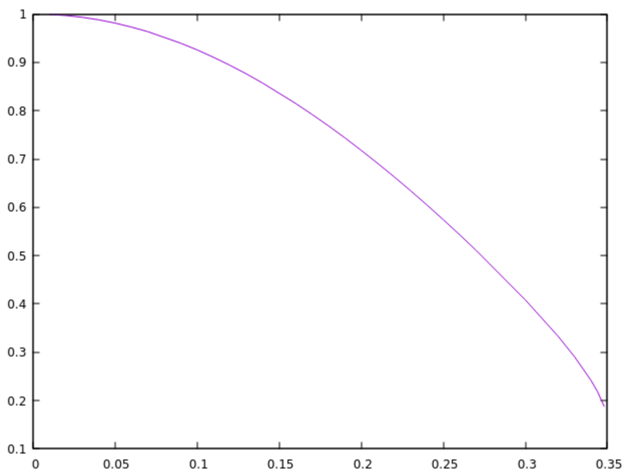

Figure 4: $\sigma(0)$ (or $F_0(0)^*$) from the coupling constant κ for $\omega = 0.1$

Conclusion

Further investigations of gauge coupled Skyrmion are to be held to get the second branch of solutions and to acquire a dependence on the gauge coupling constant g. Nevertheless, the results that were gathered from the primary calculations shows that the increasing of ω changes the view of dependence $M(\kappa)$ and increases the value of κ_{crit}.

We will continue calculations to provide more accurate analysis of the changes. The directions of future work are as follows:

1. Find the second branch of solutions for nontrivial value of coupling constant κ
2. Construct the solutions for different values of ω
3. Find critical values and determine the types of dependencies for the additional physical characteristics of the system

References

1. Shnir, Y.M., Topological and non-topological solitons in scalar field theories /Y.M. Shnir. – Cambridge: Cambridge University Press, 2018. – 280 p.

2. Manton N., Topological solitons/Nicolas Manton, Paul Sutcliffe. – Cambridge: Cambridge University Press, 2004.– 506 p.

3. M.S.Volkov and D.V.Gal'tsov, "Gravitating nonAbelian solitons and black holes with Yang-Mills fields," Phys. Rept. 319 (1999), 1-83