JOINT INSTITUTE FOR NUCLEAR RESEARCH
The Bogoliubov Laboratory of Theoretical Physics

FINAL REPORT ON THE START PROGRAMME

Second order perturbation theory in magnetohydrodynamic turbulence with parity breaking

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Abstract

We consider the stochastic model which describes developed turbulence in generalized magneto-hydrodynamics. Using standard technique, we reformulate this model as quantum field theory one. In the framework of this model we consider the vertex renormalization at two-loop approximation. For one vertex we find all diagrams which give rise to renormalization. We write a Python program that calculates a tensor structure of this diagrams. We checked Python calculations using Maple. As example, we calculate frequency and momenta integrals and find divergent part for one of this diagrams.

Introduction

Turbulence is one of the unsolved classical physics problems. Research in this area is of undoubted interest both from a theoretical and applied point of view. [1]. In this work we consider developed turbulence in generalized magnetohydrodynamic [2,3]. This describes a lot of phenomena in physics – from magnetic dynamo to astrophysical processes [4–6]. To study this regime in the magnetohydrodynamic limit [2] the following stochastic model is usually formulated [7]

$$\nabla_t \vec{v} = \nu_0 \nabla^2 \vec{v} + (\vec{b} \cdot \nabla)\vec{b} + \vec{f^v},$$
$$\nabla_t \vec{b} = \nu_0 u_0 \nabla^2 \vec{b} + A(\vec{b} \cdot \nabla)\vec{v} + \vec{f^b}. \tag{1}$$

Here $\vec{b}$ is fluctuating magnetic field, $\vec{v}$ is fluctuating velocity field, ν_0 is viscosity, u_0 is the inverse Prandtl number, A is parameter of the theory, $\vec{f^v}$ and $\vec{f^b}$ are sources of random noise. Terms with pressure fields are not given. It is assumed that the velocity and magnetic fields satisfy the incompressibility conditions. The correlation functions of $\vec{f^b}$ and $\vec{f^v}$ are given by the formulas

$$D^v_{ij}(\mathbf{x}) = \left\langle f^v_i(\mathbf{x}, t) f^v_j(0, 0)\right\rangle = \delta(t) \int \frac{d^d k}{(2\pi)^d} D_0 k^{4-d-2\varepsilon} R_{ij}(\mathbf{k}) e^{i\mathbf{kx}}, \tag{2}$$

$$D^b_{ij}(\mathbf{x}) = \left\langle f^b_i(\mathbf{x}, t) f^b_j(0, 0)\right\rangle = \delta(t) C_{ij}(|\mathbf{x}|). \tag{3}$$

Here d is dimension, $\delta(t)$ is the Dirac delta function, $k = |\mathbf{k}|$, $D_0 > 0$ is the parameter associated with the typical ultraviolet momentum scale, C_{ij} is some function whose explicit form is not important, ε is free parameter of the theory,

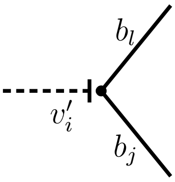
$$R_{ij}(\mathbf{k}) = \delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2} + i\rho\varepsilon_{ijl}\frac{k_l}{|\mathbf{k}|}, \tag{4}$$

ε_{ijk} is the third-rank completely antisymmetric tensor, ρ is helicity parameter ($|\rho| \leq 1$).

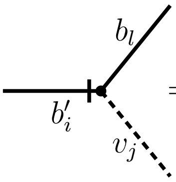
It is known that stochastic differential equations can be represented in the form of Euclidean quantum field theory [8]. For the model (1), the corresponding quantum field action is the following:

$$\mathcal{S} = \frac{1}{2}\Big(v'_i D^v_{ij} v'_j + b'_i D^b_{ij} b'_j\Big) + \vec{v'}\Big(-\nabla_t \vec{v} + \nu_0 \nabla^2 \vec{v} + (\vec{b} \cdot \nabla)\vec{b}\Big) + \vec{b'}\Big(-\nabla_t \vec{b} + \nu_0 u_0 \nabla^2 \vec{b} + A(\vec{b} \cdot \nabla)\vec{v}\Big). \tag{5}$$

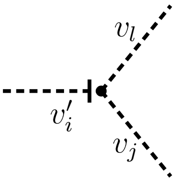
Summation over $i, j \in \{1, 2, \ldots, d\}$ is implied. The $\vec{v'}$ and $\vec{b'}$ are the additional dynamic fields. For a given quantum field theory, the following Feynman rules can be formulated in the momentum-frequency representation:



$= i\,(k_l\delta_{ij} + k_j\delta_{il}),$



$= i\,(k_j\delta_{il} - Ak_l\delta_{ij}),$

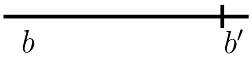


$= i\,(k_l\delta_{ij} + k_j\delta_{il}),$

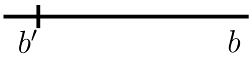
$= \frac{g_0\nu_0^3 k^{4-d-2\varepsilon}}{|-i\omega+\nu_0 k^2|^2} R_{ij}(\mathbf{k}),$

$= \frac{1}{-i\omega+\nu_0 k^2} P_{ij}(\mathbf{k}),$

$= \frac{1}{i\omega+\nu_0 k^2} P_{ij}(\mathbf{k}),$



$= \frac{1}{-i\omega+\nu_0 u_0 k^2} P_{ij}(\mathbf{k}),$



$= \frac{1}{i\omega+\nu_0 u_0 k^2} P_{ij}(\mathbf{k}).$

Here

$$P_{ij}(\mathbf{k}) = \delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}, \qquad R_{ij}(\mathbf{k}) = P_{ij}(\mathbf{k}) + i\rho\varepsilon_{ijl}\frac{k_l}{|\mathbf{k}|}. \tag{6}$$

Project goals

For this field theory we consider the renormalization of the vertex $\vec{v'} \cdot (\vec{b} \cdot \nabla)\vec{b}$ in two-loop approximation. It was necessary to determine the diagrams contributing to the renormalization, and also for one of these diagrams to calculate the tensor structure, calculate the integrals over frequencies and momenta, and find the divergent part.

Main part

The following diagrams were found to be contained in this order (see Appendix A). We calculate the following diagram

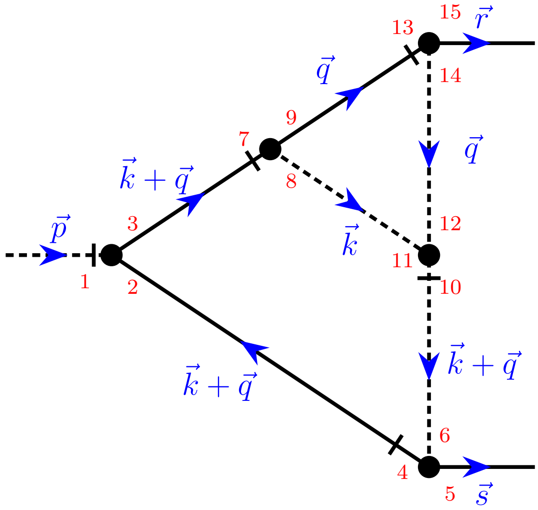


Diagram 1.

Here, the numbers denote tensor indices (it is convenient to use this notation instead of the Latin alphabet, the Einstein convention is still implied), the arrows indicate the direction of momenta, $\vec{p}$, $\vec{r}$ and $\vec{s}$ – are external momenta, $\vec{k}$ and $\vec{q}$ – are momenta over which it is necessary to integrate. The momenta $\vec{r}$ and $\vec{s}$ can be taken equal to zero. The original integral corresponding to this diagram is given by

$$\begin{aligned} I \;=\; & \frac{1}{(2\pi)^{2(d+1)}} \int d\vec{k}\, d\vec{q}\, d\omega_k\, d\omega_q \, \frac{g_0 \nu_0^3 |k|^{4-d-2\varepsilon}}{|i\omega_k + \nu_0 k^2|^2} \, \frac{g_0 \nu_0^3 |q|^{4-d-2\varepsilon}}{|i\omega_q + \nu_0 q^2|^2} \, \frac{P_{6,10}(k+q)}{i(\omega_k+\omega_q)+\nu_0(k+q)^2} \\ & \frac{P_{3,7}(k+q)}{-i(\omega_k+\omega_q)+u_0\nu_0(k+q)^2} \, \frac{P_{2,4}(k+q)}{i(\omega_k+\omega_q)+u_0\nu_0(k+q)^2} \, \frac{P_{9,13}(q)}{-i\omega_q+u_0\nu_0 q^2} \\ & [P_{8,11}(k)+i\rho H_{8,11}(k)] \; [P_{12,14}(q)+i\rho H_{12,14}(q)] \; (-i)\,[(k+q)_{11}\delta_{10,12}+(k+q)_{12}\delta_{10,11}] \\ & i\,[p_2\delta_{1,3}+p_3\delta_{1,2}] \; (-i)\,[(k+q)_6\delta_{4,5} - A(k+q)_5\delta_{4,6}] \; i\,[(k+q)_8\delta_{7,9} - A(k+q)_9\delta_{7,8}] \\ & i\,[q_{14}\delta_{13,15} - A q_{15}\delta_{13,14}] \,. \end{aligned}$$

To carry out integration over frequencies a program was written in the Wolfram Mathematica computer algebra system. To determine the tensor structure, a program was written in Python and independently verified in the Maple computer algebra system.

After calculating the tensor structure and frequency integrals I takes the form

$$\begin{aligned} I \;=\; & \frac{i\,(p_{15}\delta_{1,5}+p_5\delta_{1,15})\, g_0^2 A^2}{d(d+2)(1+u_0)} (2\pi)^{-2d} \int d\vec{k}\, d\vec{q}\, k^{2-d-2\varepsilon} q^{-d-2\varepsilon} (1-z^2)(kz-q) \\ & \left(q^3-\rho^2 k q^2 - A k^2 q + A\rho^2 k q^2\right) \left[\frac{1}{4\,(1-u_0)^{\,2}\,(k^2+q^2)^2\,(k^2+kqz+q^2)} - \right. \\ - \;& \frac{k^2+(1+u_0)q^2}{2u_0(1+u_0)\,(k^2+q^2)\,(k^2+q^2u_0)\,(k^2+2kqz+q^2)^2} + \\ + \;& \frac{u_0}{2\,(1-u_0)^{\,2}\,(k^2+q^2u_0)^{\,2}\,(k^2(1+u_0)+2kqu_0z+2q^2u_0)} - \\ - \;& \frac{u_0^2}{(1-u_0)^{\,2}\,(1+u_0)\,(k^2+q^2)^2\,(k^2(1+u_0)+2kqu_0z+q^2(1+u_0))} - \\ - \;& \frac{1}{(1-u_0)^{\,2}\,(1+u_0)\,(k^2+q^2u_0)^{\,2}\,(2k^2+2kqz+q^2(1+u_0))} + \\ + \;& \left. \frac{q^2(2k^2+(1+u_0)q^2)}{2\,(1+u_0)\,(k^2+q^2)^2\,(k^2+q^2u_0)^{\,2}\,(k^2+2kqz+q^2)} \right]. \end{aligned}$$

For integrationg over the momenta and finding divergent part we use Wolfram Mathematica. The Prandtl number in the calculation can be set equal to $u_0 \approx 1.393$ [9]. The final result is

$$I = \left(\frac{i\,(p_{15}\delta_{1,5} + p_5\delta_{1,15})\,g_0^2}{\varepsilon}\right)\times \\ \times\Big(-7.06192\cdot 10^{-7}A^2 + 4.44924\cdot 10^{-7}A^2\rho^2 + 3.47105\cdot 10^{-7}A^3 - 4.44924\cdot 10^{-7}A^3\rho^2\Big). \tag{7}$$

Results

1. A program is written in Python that performs the necessary convolutions in the tensor structure of diagrams. The results of this program have been independently verified using the Maple computer algebra system.

2. For one of the diagrams (see *Diagram 1*) integration over frequencies and momenta was carried out, and the pole part in ε was find using the Maple computer algebra system.

Conclusion

The model that describes the developed turbulence in magnetohydrodynamics is considered. This model is a system of stochastic differential equations, and, as is known, can be represented in the form of a quantum field theory. This makes it possible to apply the apparatus of the renormalization group developed in elementary particle physics and the physics of critical phenomena. We have analyzed the renormalization of the vertex $\vec{v'}\cdot(\vec{b}\cdot\nabla)\vec{b}$. Diagrams were obtained that give rise to the renormalization of a given vertex. Further, as an example, we have analyzed and calculated one of the diagrams. In the future, it is planned to calculate the rest of the diagrams given in Appendix A in a similar way, as well as write a program in the Wolfram Mathematica that automates routine calculations.

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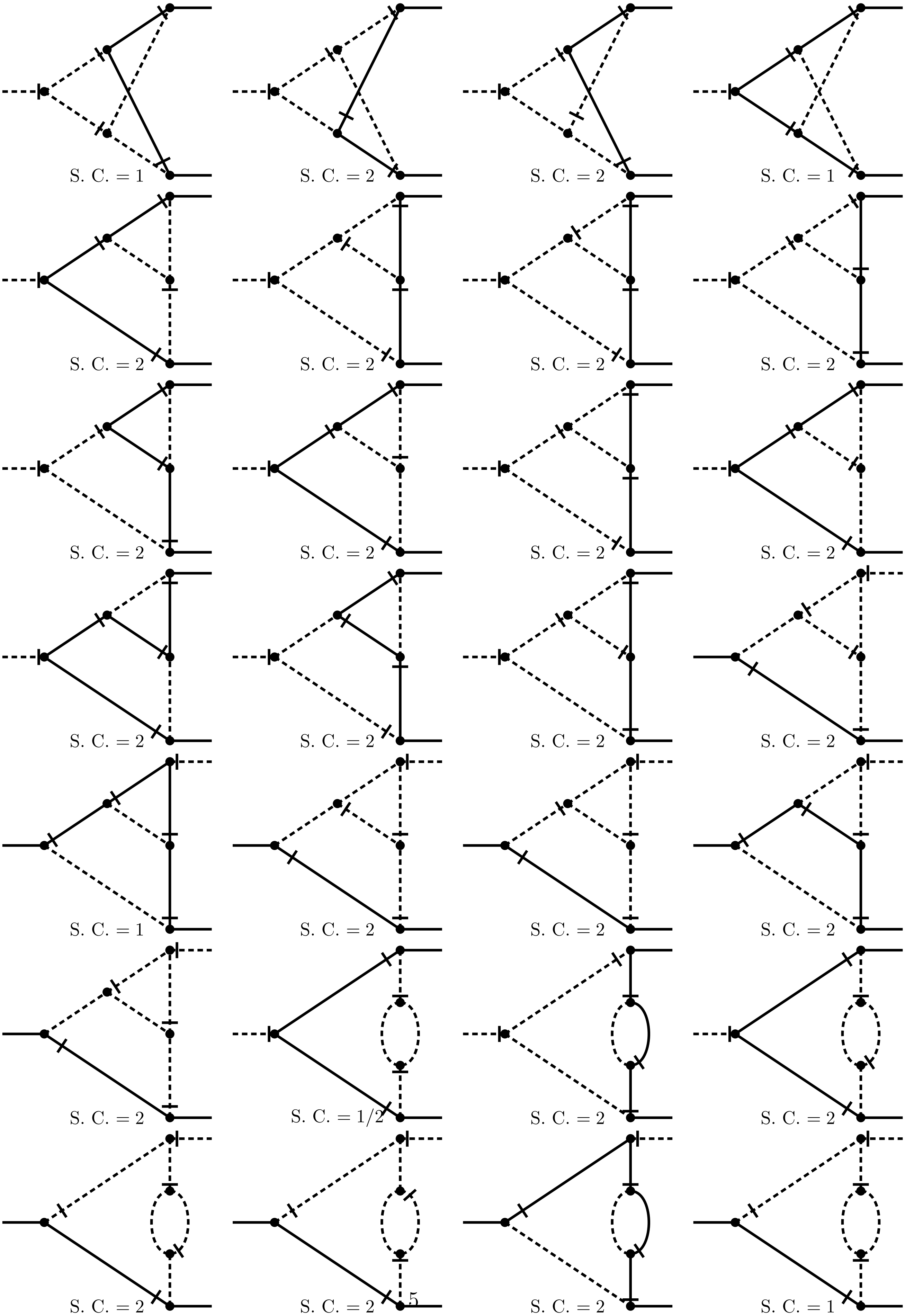
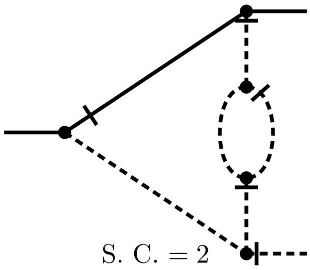
Appendix A. The two-loop diagrams which give rise to renormalization of $\vec{v'} \cdot (\vec{b} \cdot \nabla)\vec{b}$





S. C. = 2

Here S. C. is symmetry factor of corresponding diagram.

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