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Kink-antikink collisions in a model interpolating between the sine-Gordon model and ϕ^4 theory

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1 Introduction

The history of soliton started near the Edinburgh and Glasgow Union Canal in 1834. John Scott Russell, the Scottish scientist and engineer, observed the phenomenon on the surface of the water and called it " a solitary wave". He wrote:

> *I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large* ***solitary elevation****, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed*

The scientist conducted his independent laboratory experiments using a huge reservoir of water and, thus, quantitatively determined regularities of the phenomenon seen. Later, in 1895, Dutch physicists Diederick Korteweg and Gustav de Vries derived a nonlinear partial differential equation (the KdV equation) that, according to them, could describe Russell's experiments. Unfortunately their work and Russell's observations were ignored by mathematicians and physicists who studied waves of water until 1965, when Norman Zabuski and Martin Kruskal published their numerical solutions to the KdV equation. Further, they found an unexpected property of the equation: from a smooth initial waveform, waves with sharp peaks emerge. Those pulse-waves move almost independently with constant speeds and pass through each other after collisions. A detailed analysis confirmed that each pulse is a solitary wave and these solitary waves behave like stable particles. In modern physics, a suffix "on" is used to indicate the particle property, thus the solitary wave was named a "soliton".

The definition of a **soliton** is the following: it is a solution of nonlinear partial differential equation of a field theory, which is localized, keeps its localized shape over time and is preserved under interactions with other solitary waves. The special property of solitons that their stability stems from the delicate balance of "nonlinearity" and "dispersion" in the model equations. The nonlinear effect causes the steepening of waveform, while the dispersion effect makes the waveform spread.

Solitons appear in almost all branches of physics, such as hydrodynamics, plasma physics, nonlinear optics, condensed matter physics, low temperature physics, particle physics, nuclear physics, biophysics and astrophysics. The most prominent example of a soliton in nature is the Great Red Spot of Jupiter.

Since the discovery of solitary waves, a variety of localized pulses has been investigated in both one dimension and multiple spatial dimensions. It is worth noting that there are some fine points in determining what a solitary wave is in multiple spatial dimensions, that is why though some localized pulses have ”on”-ending name they do not in general have similar interaction properties as solitons.

Topological solitons occupy a special place in a huge family of solitary waves. Topological solitons are the solitons, that emerge because of topological constraints. One of the examples is a skyrmion, which is the solution of a nuclear model whose topological charge is the baryon number. The example we are interested in is a **kink**, the only one-dimensional topological solitary wave. It represents a twist in the value of a solution and causes a transition from one value to another.

Now let us consider what **intagrable systems** are. Certain partial differential equations have localized, smooth soliton solutions which do not disperse. Moreover, if a number of these solitons are superposed at large separations, and set in motion, then there is a collision, but they emerge from the collision almost unchanged. The number of solitons is unchanged, and the momenta before and after are all the same. If one could label the solitons, then one would say that the momenta had been permuted. The solitons just experience a time delay or time advance due to the collisions. Such interesting behavior occurs in integrable PDEs. The conservation of number and momenta of solitons is a consequence of an infinite number of conservation laws. Classic example of an intagrable system is a **sine-Gordon model**, which supports solutions in the form of kinks.

The Lagrangian density defining the sine-Gordon model is

$$\mathcal{L} = \frac{1}{2}\phi_t^2 - \frac{1}{2}\phi_x^2 - (1 - \cos\phi) \tag{1}$$

The sine-Gordon equation is a nonlinear hyperbolic partial differential equation in $1+1$ dimensions involving the sine of the unknown function (where the name of the model comes from):

$$\phi_{tt} - \phi_{xx} + \sin\phi = 0, \tag{2}$$

where x and t are the space-time coordinates. Obviously the zero energy vacua of this model are given by the constant solutions $\phi = 2\pi n$, where n is an integer.

Another model that supports kink solutions is a ϕ^4 model, but unlike sine-Gordon it is non-integrable. The Lagrangian density of the model is

$$\mathcal{L} = \frac{1}{2}\phi_t^2 - \frac{1}{2}\phi_x^2 - \lambda(m^2\phi^2)^2 \tag{3}$$

where m and λ are positive real constants. Global minima of the potential occurs at $\phi = m$ and $\phi = -m$. The model also can be represented by the nonlinear dispersive wave equation with a double-well potential:

$$\phi_{tt} - \phi_{xx} - 4\lambda(m^2 - \phi^2)\phi = 0, \tag{4}$$

This ϕ^4 model arises in many physical situations. For example, it has a number of applications in condensed matter physics and was applied to describe solitary waves in shape-memory alloys and to describe soliton excitations in DNA double helices in biophysics.

Collisions in nonintegrable systems can lead to a rich set of behaviors, depending on the initial paramets. For example, for a certain initial velocities in kink-antikink collisions in ϕ^4, the so-called two-bounce resonance occurs, when solitary waves begin to move apart, but then turn around and collide a second time before finally escaping from each other's influence. Moreover, there is a possibility of higher order resonance collisions, and the processes happening after collisions dependence on the initial velocities of kinks has a pronounced fractal structure. The process of kink-antikink collision in the ϕ^4 model is described in papers [3] and [4].

The aim of our work is to represent a model, interpolating between integrable sine-Gordon and non-integrable ϕ^4, find solutions in the form of kink if exist, and describe all the processes happening during the collision of kink and antikink. Obviously, the model constructed will be a non-integrable model in any case, and therefore the results will not be trivial.

2 The model

We consider the usual $1 + 1$ dimensional scalar model defined by Lagrangian density

$$\mathcal{L} = \frac{1}{2}\phi_t^2 - \frac{1}{2}\phi_x^2 - U(\phi) \tag{5}$$

with a potential

$$U(\phi) = (1 - \epsilon)(1 - \cos\phi) + \frac{\epsilon\phi^2}{8\pi^2}(\phi - 2\pi)^2 \tag{6}$$

where $\epsilon \in [0, 1]$ is a dimensionless parameter of the model. Setting $\epsilon = 0$ leads to the standard sine-Gordon potential, while setting $\epsilon = 1$ brings shifted and rescaled $\mathcal{Z}_2$ potential of the ϕ^4 model with two vacua $\phi_1 = 0$ and $\phi_2 = 2\pi$. Thus, the model (5) with the potential (6) interpolates between the sine-Gordon model and ϕ^4 theory.

An equation of motion corresponding to the model is the following:

$$\phi_{tt} - \phi_{xx} + (1 - \epsilon)\sin\phi + \frac{\epsilon\phi}{2\pi^2}(\phi - 2\pi)(\phi - \pi) = 0 \tag{7}$$

As said before there is an analytical form of the solution for the sine-Gordon and ϕ^4 models. Since we are interested in considering the model for any possible value of the parameter ϵ, we cannot use any of the analytical solutions for specifying the initial configuration of the kinks. Thus, first we need to obtain a static solution from equation

$$\phi_{xx} + (1 - \epsilon)\sin\phi + \frac{\epsilon\phi}{2\pi^2}(\phi - 2\pi)(\phi - \pi) = 0 \tag{8}$$

When a static solution in the form of a kink and an antikink is obtained it becomes possible to specify the initial conditions and to solve the equation (7).

3 Numerical results

In order to solve the problem of moving kinks we use Lorentz transformation for the space grid:

$$\delta x \to \delta x\sqrt{1-v^2} \tag{9}$$

$$\phi_x(x) \to \frac{v\phi_x(x)}{\sqrt{1-v^2}} \tag{10}$$

where δx is a spatial step, v is initial velocity of kink and antikink and $\phi(x)$ is a numerically founded static solution. After the transformation we take as initial conditions a kink with velocity v and an antikink with velocity $-v$:

$$\phi(x,0) = \phi_K(x+x_0,v) + \phi_{\bar{K}} - 2\pi \tag{11}$$

$$\dot{\phi}(x,0) = \dot{\phi}_K(x+x_0,v) + \dot{\phi}_{\bar{K}} \tag{12}$$

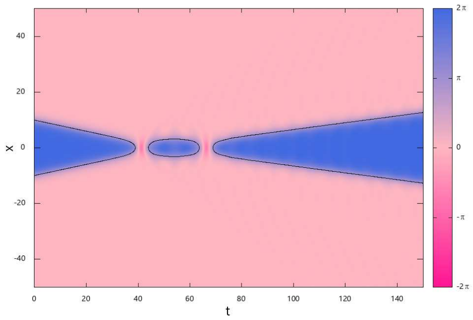
where ϕ_K and $\phi_{\bar{K}}$ is a transformed solution for kink and antikink. We fixed the initial position $x_0 = 10$, i.e., the kink solution centered at x_0 and the antikink at x_0.

To solve partial differential equation (7) we used method of lines implying a discretization over a spatial variable to get a system of ordinary differential equations. This ODE-system is solved using the eighth-order accurate Runge-Kutta method with the time step $\delta t = 0.05$. The space interval L is chosen reasonably large ($L = 200$) to avoid the influence of radiation reflected.

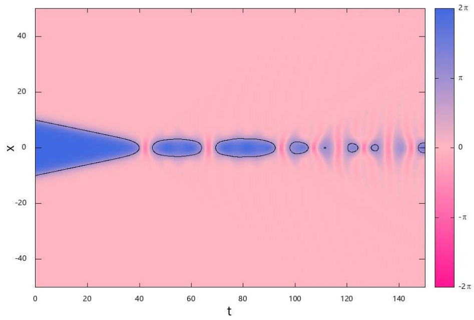
3.1 Kink-antikink collisions in the model with $\epsilon = 0$ and $\epsilon = 1$

We start our analysis by reviewing some aspects of the model, corresponding to $\epsilon = 0$ and $\epsilon = 1$, i.e., an integrable sine-Grodon model and ϕ^4 theory. For ϕ^4 model at large speeds the two waves are immediately reected upon collision, as seen in part (c) of Figure 1. Note that velocity after the collision differs from the initial velocity. Immediately reflection occurs up to a certain value of the speed v_{cr}. Below this value the picture of the collision changes. For example, kink and antikink can form into a sort of chaotic bound state, as seen in part (b) of the figure, or, what is more interesting, escape each other's influence after two or even more collisions. According to our numerical results for the model (with shifted and rescaled potential of model (3)) $v_{cr} = 0.2580$.

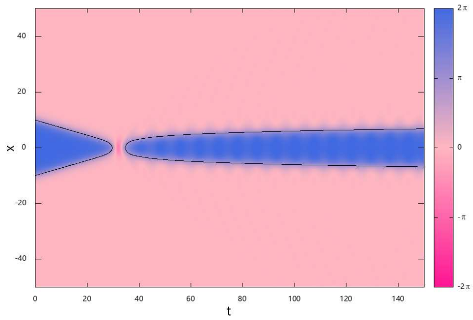
The structure of the bounces is better visualized in the Figure 2 where the field value in the center on the collision is represented. It is shown that three-bounce windows are situated at the edge of the two-bounce windows. It is possible to show the existence of windows of a higher order with the same "$(n+1)$ next to (n)" structure, so the whole frame of the collision is fractal.



(a)



(b)



(c)

Figure 1: Kink-antikink collisions in the model with $\epsilon = 1$ for different initial velocities: (1a) $v = 0.195$, (1b) $v = 0.190$ and (1c) $v = 0.260$

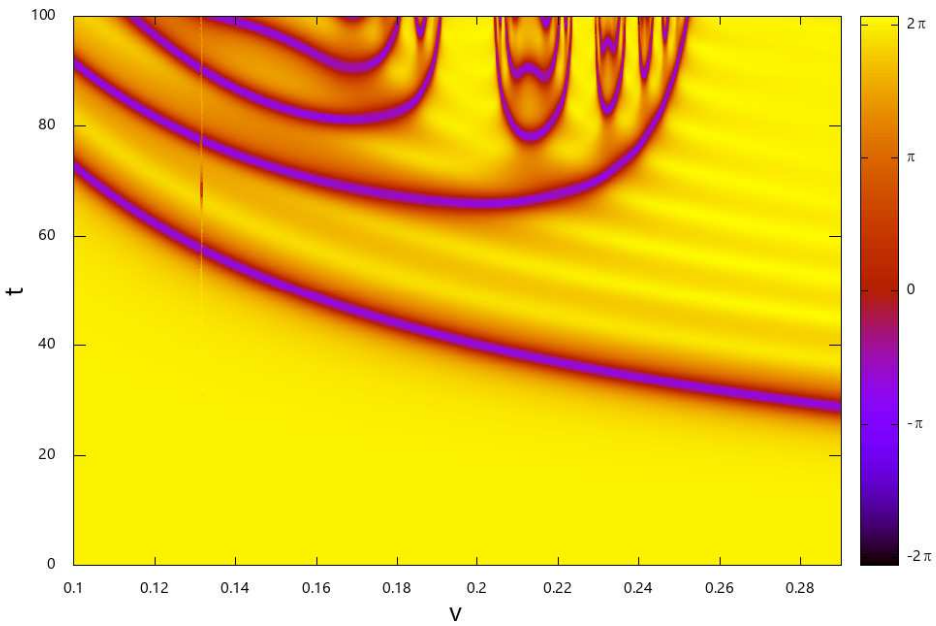


Figure 2: Field value in the center of kink-antikink collision for different initial velocities for $\epsilon = 1$.

The picture of the collision in the sine-Gordon is much simpler as expected. Field value in the center is abruptly changing, kink and antikink are scattering, no longer interacting. Note, that this happens for any initial velocity.

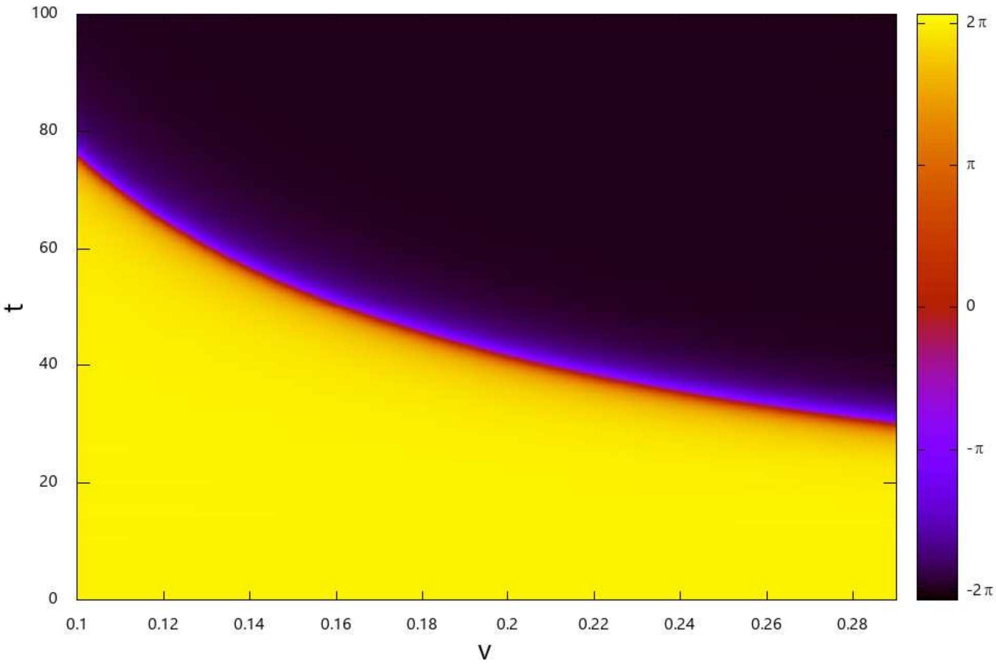
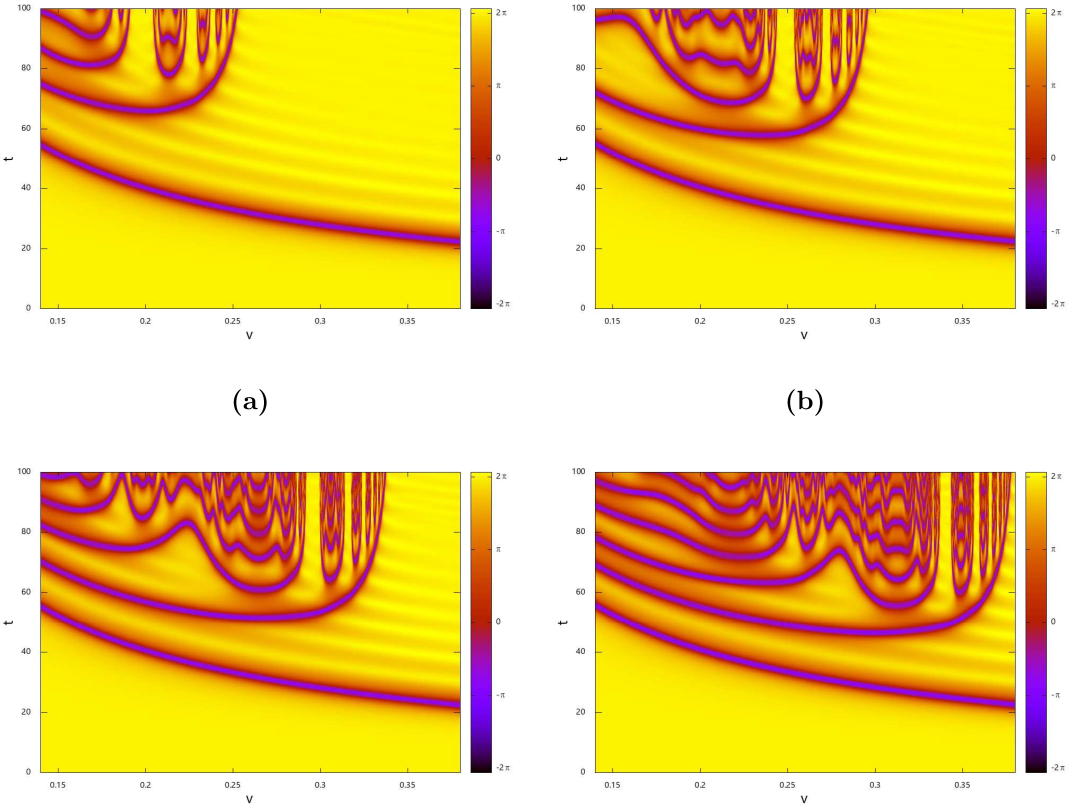


Figure 3: Field value in the center of kink-antikink collision for different initial velocities for $\epsilon = 0$.

3.2 Kink-antikink collisions in the model with $\epsilon \in (0, 1)$

Let us now consider kink-antikink collisions in the model with noncritical values of ϵ. As an example we present distributions of field value in the center of collision for a number of values of ϵ on Fig. 4. One readily observes that as ϵ decreases from one the critical value of kink velocity $v_{\rm cr}$ increases, while multibounce windows get more narrow and shift to higher values of velocity. For each n there is critical value of $\epsilon_{{\rm cr}_n} > 0$ below which n-bounce window ceases to exist. As ϵ approaches $\epsilon_{{\rm cr}_n} > 0$ n-bounce window gets thinner until it vanishes at critical value.



(a) **(b)**

(c) **(d)**

Figure 4: Field value in the center of kink-antikink collision for a set of values of ϵ.

4 Conclusions

We have investigated $1+1$-dimensional scalar field theory interpolating between well known sine-Gordon and ϕ^4 models. We observed strong evidences for existence of topological solitons - kinks in whole range of defining parameter of model, ϵ, and studied them numerically. We investigated kink-antikink collisions in the model for whole range of parameter ϵ and found that properties of collisions again interpolate between those in sine-Gordon and ϕ^4 models. The work is not finished yet: in future we are aiming to better understand properties of kink-antikink collision in model in proximity of limiting values of parameter ϵ and investigate the stability of solutions found.

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References

[1] Nicholas Manton and Paul Sutcliffe. *Topological Solitons.* Cambridge University Press, 2004

[2] Yakov M. Shnir. *Topological and Non-Topological Solitons in Scalar Field Theories.* Cambridge University Press, 2018

[3] Roy H. Goodman and Richard Haberman. *Kink-Antikink Collisions in the* ϕ^4 *Equation: The n-Bounce Resonance and the Separatrix Map.* SIAM J. APPLIED DYNAMICAL SYSTEM, 2005

[4] D.K. Campbell, J.F. Schonfeld and C.A. Wingate. *Resonance structure in kink-antikink interactions in* ϕ^4 *theory.* Physica 9D, 1983

[5] P.G. Drazin and R.S. Johnson. *Solitons: an introduction.* Cambridge University Press, 1989.